COMPAS MASONRY: A COMPUTATIONAL FRAMEWORK FOR PRACTICAL ASSESSMENT OF UNREINFORCED MASONRY STRUCTURES

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Abstract. *In recent years, our (academic/theoretical) understanding of the behaviour of unreinforced masonry (URM) structures has improved significantly, and many advanced technological solutions for conservation have been developed. However, there is still a lack of appropriate methods and tools that can be used for the assessment of URM structures in every day practice. Therefore, since 2018, the Block Research Group has been working on "Practical Stability Assessment Strategies for Vaulted Unreinforced Masonry Structures" with support of the Swiss National Science Foundation (SNSF). The goal of this research project is to create tools suitable for everyday engineering practice and to develop appropriate analysis strategies for diverse contexts and circumstances related to the availability of time, budget and available data. The main outcome is COMPAS Masonry: an open-source, Python-based computational framework for the assessment of URM structures. It provides a general purpose toolbox for working with assemblies (compas_dem) and three custom made open-access solvers that can deal with different aspects of the assessment of masonry structures: compas_tna based on Thrust Network Analysis, compas_prd based on the Piecewise Rigid Displacement method, and compas_rbe based on the Rigid Block Equilibrium.*

1 INTRODUCTION

Unreinforced masonry (URM) is one of humankind's longest-lasting construction methods and forms the structural basis for most of the residential buildings all over the world. The assessment of URM structures is an atypical mechanical problem: constructions are comprised of individual discrete elements; large displacements and deformations are common, and, particularly for historic buildings, material properties and boundary conditions are unknown or unknowable.

Many of the structural analysis tools/software available today were developed for entirely different structural systems with very different materials such as steel, concrete and timber, and should therefore not be applied to the analysis of masonry structures. Indeed, it is well known that some well-established engineering methods, such as Finite Element Analysis (FEA), do not apply to the assessment of masonry because of their numerical inability to deal with the intrinsic unilateral behaviour of masonry: many of the required mechanical parameters are unknown or even unknowable, and zero-energy modes (ill-conditioned stiffness matrices) related to fractures cannot be taken into account. The reader is referred to e.g. [1, 2], where these aspects have been highlighted. Moreover, even technical regulations are not specifically conceived for URM structures, leaving practitioners the freedom to utilise any method at their disposal. If the masonry knowledge of the practitioner is not in-depth and accurate, the risk of tackling the assessment with non-appropriate tools, and then to wrongly intervene, becomes a certainty. One of the consequences is that many restorations actually compromise the mechanical behaviour of structures that had been standing for centuries.

In 1966, Jacques Heyman [3] gave the theoretical basis to apply Limit Analysis to masonry structures. Still today, his limit analysis theory based on three crude material assumptions, is recognised as one of the best approaches for the assessment of URM structures. Nevertheless, even though the Limit Analysis theory is widely accepted, there is still a lack of user-friendly numerical implementations that can be used by specialists in their everyday activities.

More recently, Discrete Element Modelling (DEM) methods, in particular 3DEC by Itasca [4], have also been successfully applied to the assessment of 3D masonry structures. Currently, they represent the unique alternative to the standard engineering tools conceived for other materials. Unfortunately, since most implementations are only available through (often expensive) proprietary software their use is mainly restricted to academic research.

Since 2018, with the support of the Swiss National Science Foundation (SNSF), the Block Research Group has been developing COMPAS Masonry, a computational framework for "Practical Stability Assessment Strategies for Vaulted Unreinforced Masonry Structures". The goal is to develop better approaches and tools for engineering practice as it relates to the analysis of URM structures.

COMPAS Masonry bundles four Python-based tools: *compas_dem, compas_tna*, *compas_prd*, and *compas_rbe*. The proposed computational framework allows managing complex 3D geometries easily, to work with different loading conditions, to take into account the effects of (large) foundation displacements and, when needed, also mechanical and geometrical imperfections. Furthermore, by making these tools available, this project provides a framework for developing a better understanding of the behaviour of masonry structures which will reflect into more appropriate and less intrusive restoration interventions.

2 COMPUTATIONAL FRAMEWORK

COMPAS Masonry is an open-source, Python-based computational framework for the assessment of URM structures. COMPAS Masonry bundles specific open-access, Python packages and provides benchmark data, protocols and procedures for performing analyses on real-world masonry problems using any combination of the available computational approaches. COMPAS Masonry includes a general purpose toolbox for working with assemblies (*compas_dem*), and three custom solvers that can deal with different aspects of the assessment of masonry: *compas_tna* [5], based on the Thrust Network Analysis (TNA) by Block [6]; *compas_prd* [7], based on the Piecewise Rigid Displacement (PRD) method by

Iannuzzo [8]; and *compas_rbe* which is based on the Rigid Block Equilibrium (RBE) by Withing [9]. *compas_dem* [10] can be used in combination with any of the custom solvers as well as with the commercial DEM solver 3DEC by Itasca [4].

Table 1: Overview of the models, approaches and the numerical procedures adopted in COMPAS Masonry.

Tool	Model	Approach	Solution
3DEC	DEM	Netwton's laws	Explicit dynamics
compas_tna	Heyman	Equilibrium	Nonlinear optimization
compas_prd	Heyman	Energy	Linear optimization
compas_rbe	Livesley (extended)	Equilibrium	Quadratic optimization

As one can see from **Table 1**, including 3DEC, these tools are based on different models/approaches. Specifically, 3DEC uses an explicit dynamics approach, *compas_tna* and *compas_prd* are Limit Analysis-based methods framed within the Heyman's model [3] and *compas_rbe* is based on an extension of Livesley's equilibrium approach [11]. The diverse approaches reflect into different numerical procedures adopted to find solutions: three of them are based on optimisation procedures (linear and nonlinear programming) while 3DEC integrates Newton's second law of motion using the central finite difference method with respect to time.

2.1 General purpose toolbox: *compas_dem*

compas_dem provides tools for the generation of assemblies of discrete elements and especially for the management of relationships between the individual parts. *compas_dem* provides several purposes tools: to generate either parametric or measured 3D geometries; to model/generate mechanical and geometrical imperfections; to detect interfaces in the assembly; to visualize interface forces. Moreover, it provides a common interface for different solvers in the background and tools to post-process and visualise their results.

2.2 Discrete Element Modelling: 3DEC

One of the solvers that can be used in combination with *compas_dem* is 3DEC by Itasca. 3DEC is commercial/proprietary Discrete Element Modelling software, and it is used in this project to benchmark and guide the development of the three other COMPAS Masonry tools. Compared to traditional structural analysis tools DEM software has three main peculiarities: the analysis model consists of separate blocks that can move and deform independently; large displacements are possible; blocks can detach from each other, and new contacts can form. In 3DEC, both rigid and deformable blocks can be considered, and unilateral contact conditions can be defined. In particular, using rigid blocks acting unilaterally and a Mohr-Coulomb criterion, the only mechanical parameters required are: material density, friction angle, Young's and shear moduli (to evaluate the joint stiffnesses). The calibration of these parameters is crucial and, for this purpose, in this project, tests on physical models are performed to calibrate 3DEC parameters and to validate its results [12, 13, 14]. Once the calibration of the mechanical parameters is done, 3DEC represents a reliable tool for the assessment of URM structures, and it can be used for many assessment problems: stability in a given configuration, load-bearing capacity, displacement capacity, dynamic loading conditions, analysis of the settlements. The downside of the numerical approach (i.e. explicit dynamics, see **Table 1**) is that the computational time needed to solve some typical problems can range from minutes for very simple problems to several hours for problems with more realistic complexity.

2.3 Thrust network analysis (TNA): *compas_tna*

compas_tna provides a base implementation of Thrust Network Analysis (TNA) by Block [6, 15]. TNA works by computing an equilibrated, compressive and admissible thrust network entirely contained within the structural geometry of the masonry. This network is the representation of a one-dimensional, singular, compressive stress field in which the forces are concentrated on the edges and the loads and restraints are applied in the nodes. The method is a direct application of the safe theorem of limit analysis by Heyman [3, 16] since the admissible thrust network corresponds to a lower-bound solution of the limit state of the structure. *compas_tna* sets up and solves a constrained nonlinear optimisation problem [5] searching for a particular stress state of the structure by tuning the objective function and the constraints. Several objective functions are implemented to face different structural problems, such as the search for the minimum and/or maximum thrust and maximum collapse load multiplier. The main constraints of the optimisation problem enforce the heights of the network nodes to be contained within the structural geometry represented by its internal and external surfaces. Additional constraints can also be included to simulate the conditions observed on existing structures, such as simulation of cracks, limits on the reactions of the vault, and openings. By searching among different stress states, TNA can also be used to obtain an estimate of the level of stability of masonry vaults.

2.4 Piecewise rigid displacement (PRD): *compas_prd*

compas_prd is a new computational tool that stems from the piecewise rigid displacement (PRD) method [8]. With the PRD method, a masonry structure is modelled as composed of normal, rigid, no-tension (NRNT) material, which mathematically frames into continuum mechanics Heyman's material assumptions. With PRD both mechanisms and internal forces can be found simultaneously by solving two dual linear programming (LP) problems [17]. The primal LP problem, representing the minimum of the total potential energy in the space of piecewise rigid displacements, returns a rigid macro-blocks partition of the structural domain and thus cracks between adjacent elements, representing singular strain fields. The dual LP problem, that is the minimum of the complementary energy, returns internal and external forces in equilibrium with the external loads and compatible with the crack pattern solving the primal problem. Different mechanical problems can be tackled: stability in the initial configuration, effects of foundation displacements [18]; assessment of the safety under horizontal actions [19]; and, the effects of large foundation displacements [20].

2.5 Rigid block equilibrium (RBE) method: *compas_rbe*

compas_rbe provides a base implementation of Rigid block equilibrium (RBE) by Withing, which is a numerical method for the three-dimensional analysis of equilibrium states of URM structures modelled as assemblies of distinct rigid blocks. The research of Livesley [11, 21] constitutes one of the basic studies to address RBE. Livesley was the first to solve the equilibrium and mechanism formulations of Limit Analysis using linear programming, and he was also the first to attempt to introduce the mechanism of sliding at the joints. This issue was solved by Gilbert and Melbourne [22], who implemented sliding between joints and incorporated friction into the equilibrium equations. In *compas_rbe* the blocks are considered to be infinitely rigid and to have infinite compressive strength. Interfaces between the blocks are considered to have a finite frictional capacity. *compas_rbe* provides the necessary contact forces (i.e. compression, tension and friction forces) for the assembly to be in equilibrium under given external loads. Furthermore, it extends Livesly's formulation enlarging the space of stress solutions by considering tensile forces in penalty formulation.

3 MODELS AND ASSESSMENT TOOLS: INPUT AND OUTPUT

Master builders constructed masonry not just as chaotic union of bricks but rather as a smart and appropriate assemblage of blocks that prevents sliding failures by ensuring proper connections in each element and in between adjacent elements ("rules of art") [3, 23, 24, 25]. When a masonry construction is structurally sound, it behaves as a whole, its stability depends on its geometry [3, 26, 27, 28, 29] and it can accommodate small external changes with its nucleation into rigid macro-blocks, visible through the fracture pattern. This peculiar behaviour allows masonry structures to be flexible, exhibiting a ductile (elastic) response to the external changes [30].

To understand if the structure to assess responds to this criterium, it is crucial to know some typical aspects: materials and construction phases of the building, history (seismic events, collapses, reconstructions, additions, etc.), stereotomy, construction techniques, and a proper investigation of the geometry. A particular attention needs to be paid to a the exact definition of the relevant crack pattern and measurements of the leaning of the walls and columns. This information is central for understanding the global behaviour of the structure, where each element (e.g. vaults, arches, walls, columns, buttresses, flying buttresses) is a part of a complex mechanical system. After this evaluation, which demands an in-depth knowledge of the specialist, appropriate assessment strategies can be defined and specific tools can be adopted.

3.1 Simplified models

If the structure is structurally sound (*rules of art* [24]), simplified models based on rigid-plastic constitutive relations can be properly adopted [31]. Two of the three solvers available in COMPAS Masonry are based on Heyman's model: *compas_tna* and *compas_prd*. The Heyman model is very crude (no tensile strength, infinite compressive strength, no-sliding failures) not requiring any mechanical parameter but, it represents a robust approach to account for the peculiarities of masonry structures: geometrical stability, unilateral behaviour, fractures, nucleation of the domain into rigid macro-blocks. *compas_rbe*, based on Livesley's formulation, represents an extension of Heyman's model and it requires only one mechanical parameter: the friction angle. Therefore, the solvers of COMPAS Masonry provide a simplified approach to assessment of URM structures that does not require all information about the structure to be known. In the next sections, we highlight, the input data required and the output provided by each COMPAS Masonry tool (see **Table 2**, **Table 3**)

Table 2: Input data required by COMPAS Masonry tools and 3DEC.

Tool	Suitable for	Input	Imperfections	Friction	Other mechanical parameters
3DEC	3D geometry	Stereotomy	✓	✓	✓
compas_tna	3D vaulted structures	External and internal surfaces	✗	✗	✗
compas_prd	2.5D geometry	Geometry or stereotomy	✗	✗	✗
compas_rbe	3D geometry	Stereotomy	✓	✓	✗

The significant benefit in adopting the simplified models provided in COMPAS Masonry is that, beyond accounting for the peculiarities of the masonry with a very crude mechanical characterisation, they provide fast computational solving. In particular, it is well known that for huge mathematical problems (e.g. thousands of unknowns), the linear programming framework offers algorithms able to find a solution in a few seconds. Therefore, solvers based on linear programming (such as *compas_prd*) are useful when we have to face large problems or to approach inverse analysis, where a wide spectrum of solutions is required. Quadratic and, in general, nonlinear optimisation procedures (*compas_tna* and *compas_rbe*) require longer computational time which can range from a few seconds to several minutes. The fast computational solving of the COMPAS Masonry tools constitutes a big advantage over DEM analyses (e.g. 3DEC).

Table 3: Output data provided by COMPAS Masonry tools and 3DEC.

Tool	Forces	Min/Max Thrusts	Displacements	Cracks/ Mechanisms
3DEC	✓	✗	✓	✓
compas_tna	✓	✓	✗	✗
compas_prd	✓	✗	✓	✓
compas_rbe	✓	✗	✗	✗

3.2 *compas_tna*: input/output

compas_tna is especially useful to analyse 3D complex vaulted structures (see **Table 2**). It needs as an input only the internal and external surfaces describing the geometry of the vault. Since it searches for admissible stress solutions for the structure, the output is an equilibrated thrust network of compressive forces entirely contained within the structural geometry. *compas_tna* can directly provide the minimum and maximum thrust exerted by the structure.

3.3 *compas_prd*: input/output

compas_prd it is a direct kinematic-based approach and stands on two dual energy criteria. Currently, it is implemented for 2.5D geometries, that is: the analysis is planar but non-uniform, symmetrical orthogonal depths can be considered. The input required is a discretised geometry of the structure (see **Table 2**). In some cases, the discretisation can coincide with the structural stereotomy while in other cases, the discretisation has to be handled accordingly [32]. Its solutions consist of displacements (i.e. potential mechanisms and cracks) and corresponding internal forces. (see **Table 3**).

3.4 *compas_rbe*: input/output

compas_rbe solves the equilibrium problem for 3D assemblies of rigid, distinct blocks. As input, it requires the stereotomy of the structure and the values describing the friction between the blocks. It can handle geometrical and mechanical imperfections (random distribution of the friction angle), allowing an in-depth investigation of their role in the global mechanical behaviour (see **Table 2**). As solutions, it provides the internal stress states represented by forces acting on the interface between two adjacent blocks. The solution can also involve tensile forces in a penalty formulation and friction forces higher than the one permitted by the actual friction values. This peculiar feature enables the user to understand the main critical regions of the assembly where sliding or detachments can occur.

4 COMPUTATIONAL ANALYSES SCENARIOS

In this section, we discuss the main problems concerning the assessment of historic masonry structures and how they can be tackled within COMPAS Masonry. When we have to assess the safety of masonry buildings, the typical issues are: stability in the reference configuration; load-bearing capacity; effects of settlements; displacement capacity. Horizontal actions (e.g. seismic actions or wind) deserves a special treatment even if they can be framed as a load-bearing capacity case. In **Table 4**, a schematic overview of the assessment problems that can be tackled by each tool is presented.

Table 4: Overview of the problems tackled by COMPAS Masonry tools.

Tool	Stability in ref. configuration	Load-bearing capacity	Horizontal forces	Effects of settlements	Displacement capacity
3DEC	✓	✓	✓	✓	✓
compas_tna	✓	✓	✓	✗	✗
compas_prd	✓	✓	✓	✓	✓
compas_rbe	✓	✓	✓	✗	✗

4.1 Stability in the reference configuration

Assess if a masonry structure is stable in its reference configuration represents one of the most frequent structural problems. A widely accepted approach consists in applying the **Safe Theorem**, that is, to find a compressive stress state lying within the structural geometry and in equilibrium with the external loads. All the COMPAS Masonry tools are able to address this problem and can explore a wide range of admissible stress states either adopting different objective functions (*compas_tna*) or by adding some further constraints to the initial optimisation problem (*compas_prd* and *compas_rbe*). Outcome of a stability analysis is the safety factor. This number can be defined as an appropriate ratio between a critical position of the flow of forces and an optimal one. All the tools allow the calculation of safety factors in a given configuration by using different approaches.

Anyway, the scenario can get complex if a non-negligible crack pattern is present. This does not constitute an issue for the COMPAS Masonry tools. Indeed, the revealed fractures can be taken into account easily by each tool. In particular, while *compas_prd*, *compas_rbe* can take into account these fractures considering directly a non-perfect initial geometry, *compas_tna* offers robust algorithms to constraint the thrust networks to go throughout certain curves modelling the fracture pattern.

4.2 Load bearing capacity

Every time we are facing the problem of assessing what is the maximum load that a structure can sustain, we are solving a load-bearing capacity problem. Each tool can address this problem. In particular, the approach adopted by the three tools (*compas_tna*, *compas_prd*, *compas_rbe*) is to increment the load until a safe solution can be still found. This procedure gives rise to a set of sequential optimisation (linear and nonlinear) problems. When the load reaches its maximum allowable value, no one solution can be found anymore (for *compas_rbe*, a solution is still possible, but it involves tensile forces). In this way, we obtain the maximum allowable load and its corresponding scale factor. In this last case, *compas_prd* returns also the mechanism and corresponding crack pattern.

4.3 Load bearing capacity, a special case: horizontal forces

Horizontal forces represent a special load-bearing capacity case. When we have to account for the effects of the wind and/or of the seismic actions, one way is to model these actions by considering static, equivalent, horizontal forces All the COMPAS Masonry tools can model horizontal actions as statical forces and can evaluate the horizontal static multiplier following the same procedure exposed in **Section 4.2**.

4.4 Effects of settlements

The effects of settlements represent one of the key issues in the assessment of masonry structures. Most of the time, the crack pattern observed is due to the settlements rather than to overloading. Indeed, in most of the cases these small settlements are due to changes in the boundary conditions (e.g. a change in the thrust's inclination exerted by a pillar can reflect into small foundation displacements). A masonry structure accepts these new boundary conditions through the formation of a crack pattern. These cracks define a decomposition of the structural domain into rigid macro-blocks which allows the structure to displace almost piecewise rigidly. If the settlements are small and cannot grow up anymore, the structure is still stable. *compas_prd* allows for addressing the effects of foundation settlements (see **Table 4**). Furthermore, understanding exactly the causes (e.g. find the shape of the foundation displacements) producing a given crack pattern corresponds to an inverse analysis. Since, *compas_prd* can fastly compute a wide range of solutions, it can also be used for conducting this inverse analyse as in [32].

4.5 Displacement capacity

In some cases, foundation displacements can increase affecting strongly the (geometrical) stability of the structure. In some assessment scenarios, it is crucial to know the maximum allowable displacement value for which the structure is still stable. This demands a model which can take into account the evolution of the crack pattern/mechanism used by the structure to accommodate the increasing settlement. In this case, it is important to define a safety factor (in terms of displacement capacity) measuring how far the current structural geometry is from an unstable configuration. *compas_prd* can perform this analysis (see **Table 4**), also accounting for the evolution of the mechanism during the motion (see [20], where it was benchmarked with 3DEC).

5 CONCLUSIONS

In this paper we present an overview of the project in development since 2018 at the Block Research Group entitled "Practical Stability Assessment Strategies for Vaulted Unreinforced Masonry Structures", funded by Swiss National Science Foundation (SNSF). The aim of this project is to provide adequate tools for everyday engineering practice. In particular, one general purpose package (*compas_dem*) and three stand-alone tools (*compas_tna*, *compas_prd* and *compas_rbe*) are provided. The results of this work are shared as COMPAS Masonry, a new open-source, Python-based, computational framework for the assessment of URM structures.

This work shows the potentialities, peculiarities, limitations and applicability range of each tool, referring to specific input data (e.g. geometry, stereotomy, mechanical parameters) and to typical assessment problems (e.g. stability in the reference configuration, load-bearing capacity, effects of settlements and displacement capacity). The individual tools are designed to be as flexible as possible and to provide a simple way to manage complex mechanical and geometrical data.

Beyond benchmark data, protocols and numerical procedures, COMPAS Masonry provides a framework where these four tools can be integrated and applied in different assessment scenarios depending on the availability of information, time and budget. It also provides an easy-to-use mechanical approach, not requiring a detailed mechanical characterisation (zero mechanical parameters for *compas_tna* and *compas_prd*) and offers robust approaches and fast computational solving to account for the peculiarities of URM structures in different assessment scenarios.

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