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Area-controlled construction of global force polyhedra

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Abstract
This paper presents a method for constructing global force polyhedra with target face areas. Although the geometric construction and transformations of global force polyhedra and the generation of the corresponding form diagram have been demonstrated before, the initial design explorations using polyhedral reciprocal diagrams have been based strictly on geometric manipulations devoid of any means to explicitly control the face areas of force polyhedra, or equivalently the force magnitudes in the structure. As a result, polyhedral reciprocal diagrams have largely remained a shape exploration tool for spatial structures without the ability to incorporate quantitative, force-based constraints. The presented method significantly improves the designer's ability to control the face areas during the construction of global force polyhedra, and extends polyhedral reciprocal diagrams as a viable tool during early stages of design not just for unconstrained form-finding explorations, but also for addressing more explicit and quantitative boundary condition criteria.

Keywords: 3D graphic statics, reciprocal polyhedral diagrams, global force polyhedron, polyhedral reconstruction, boundary conditions

1. Introduction
The geometrical construction of a resultant force and the establishment of global equilibrium with respect to the boundary conditions is one of the most fundamental principles of *graphic statics*. In 2D graphic statics, boundary conditions (i.e. external loads and support locations) are typically addressed using trial funicular construction, and the procedures of this technique are well documented in numerous publications (Bow [7], Cremona [8], Wolfe [21], Allen and Zalewski [6]). Akbarzadeh et al. [3] provided a three-dimensional equivalent of the trial funicular method for constructing the *global force polyhedron* (GFP). By translating well established concepts and procedural techniques from 2D to 3D graphic statics, it was shown how a GFP can be constructed for a given set of boundary conditions. However, this method is strictly dependent on geometric procedures, which results in two key limitations.

First, the method has been applied only to determinate systems of forces (i.e. tetrahedron). In general, the method can be used to construct only one specific GFP (up to a scaling factor) for the corresponding system of forces, and it is unclear how the magnitudes of the reaction forces can be controlled and logically distributed.

Second, procedural geometric construction of the GFP means that any change in the boundary conditions requires a complete reconstruction of the polyhedral geometry. Although the step-by-step procedure is necessary and important for teaching the principles of polyhedral reciprocal diagrams, repeated reconstruction is cumbersome and inconvenient during early stages of design when the boundary conditions may not yet be finalised and multiple scenarios must still be investigated.

By using an iterative geometric solver, the construction of the GFP can be automated to negate the need for constant redraw while maintaining a flexible datastructure. Building upon the previous implementations (Ikeuchi [12], Little [16], Moni[18]) that introduced various methods for computing the geometry of polyhedra from its face areas and orientations, the method presented in this paper significantly expands the capacity of reciprocal polyhedral diagrams to address more quantitative, force-driven boundary constraints in an interactive environment where various constraints can be easily imposed.

In Section 2, we give a brief overview of the theoretical background of polyhedral reconstruction from prescribed face areas and orientations, in addition to a condensed description of the iterative solving algorithm. In Section 3, we present a simple loading scenario that has more than three supports, and how the geometry of the GFP can be freely manipulated depending on various support constraints.

2. Polyhedral reconstruction from face areas and orientations

2.1. Theoretical background

Polyhedral reconstruction is a well-researched topic in numerous fields such as computer vision, combinatorics and computational geometry. Various properties of polyhedra (i.e. vertex locations, edge lengths, face geometry, face orientations, face areas, dihedral angles, etc.) in combination with some additional metric can be used to reconstruct the geometry of polyhedra (Demaine and O'Rourke [9]). The method that is most relevant to 3D graphic statics, is the reconstruction of polyhedra from face areas and orientations (or equivalently, the magnitudes and directions of the forces, respectively).

This method originates from Minkowski's theorem on the existence of a unique polyhedron with prescribed face orientations and areas [17]. Alexandrov's version of the theorem [5] is recited below, with slightly modified notations to stay consistent with previous publications on 3D graphic statics by the authors.

If $\hat{\mathbf{n}}_0, \dots, \hat{\mathbf{n}}_\mathrm{m}$ are non-coplanar unit vectors and $A_0, \dots, A_\mathrm{m}$ are positive numbers such that

$$\sum_{\mathrm{j}=0}^{\mathrm{m}} A_\mathrm{j} \cdot \hat{\mathbf{n}}_\mathrm{j} = 0\,, \tag{1}$$

then there exists a closed convex polyhedron whose faces have outward normals $\hat{\mathbf{n}}_\mathrm{j}$ and areas A_j.

The proof for this theorem can be found in numerous texts in the literature (Grünbaum [10], Alexandrov [5], O'Rourke [19]), and therefore need not be reiterated here. Grünbaum's interpretation of equation (1) is phrased in terms of "fully equilibrated" vectors (Grünbaum, p. 332 [10]), which is directly compatible with the principles of graphic statics [1]. Although it can be mathematically proven that a unique polyhedron exists with prescribed face orientations and areas, the geometric construction procedure was never explicitly mentioned or developed.

A construction method for Rankine 3D reciprocal diagrams using conic sections and Poncelet duality has been recently introduced, demonstrating clear geometric relationships between n-dimensional reciprocal diagrams and their dual (n+1)-dimensional Airy stress functions (Konstantatou and McRobie [13]). However, this technique is primarily concerned with computing and visualising the geometry of the reciprocal polyhedra, and the ability to control the face areas of polyhedra remains unaddressed.

2.3. Computational Setup

This section provides a short background on previous implementations of polyhedral reconstruction from face areas and orientations, and the descriptions of the computational setup and procedure of the presented method.

2.3.1. Previous work

It is not until the 1980s that this problem was able to be addressed with adequate computational and numerical optimisation tools. Ikeuchi [12] first introduced a reconstruction method by using a constrained minimisation procedure. Little [16] improved this method by proposing an iterative minimisation solver using the *Extended Gaussian Image* (EGI) [11], followed by Moni's [18] introduction of the zero-area faces to address potential cross-adjacencies, and Xu and Suk's [22] use of hierarchical EGI to reconstruct concave polyhedra. In these implementations, the methods used were validated through simple examples, but the complexity and diversity of polyhedra that can be reconstructed was not demonstrated through any large sampling of convincing examples.

Lachand-Robert and Oudet [13] combined numerical and geometrical algorithms to develop a variant of a convex hull method that could reconstruct a polyhedron with 1000 given face orientations and areas. A powerful numerical solver is necessary for processing a large data set, but it has limited usefulness within the context of structural design using polyhedral reciprocal diagrams, where the user's ability to customize and interact with the method is far more important.

2.3.2. Setup and solving procedure

In essence, the iterative solving algorithm takes a set of forces (some or all magnitudes are prescribed) and reconstructs the polyhedron through an iterative geometric solver. First step of the algorithm constructs the EGI from the magnitudes and orientations of the forces. With each node of the EGI representing a face of the polyhedron (Figure 1c), arcs can be drawn on the unit sphere of the EGI to determine the main face adjacencies or connectivities. If any of these arcs are intersecting, the intersection point represents the location of a fictitious face with a target area of zero, which eventually collapses to an edge or vertex of the polyhedron [18]. The purpose of these zero-area faces is to facilitate the iterative solver by keeping track of all possible adjacencies in the datastructure regardless of the geometrical configuration of the polyhedron at the current iteration.

For example, in Figure 1e, vertex 6 of the EGI represents a point of cross-adjacency, where any combination of the faces 1, 2, 3, and 4 may be adjacent to one another depending on their target areas. Once all the zero-area faces have been found and located, a unit polyhedron can be easily constructed which has the dual datastructure of the EGI. Finally, the algorithm takes this unit polyhedron and scales each face independently from one another towards its prescribed target area, while updating both the EGI and the geometry of the polyhedron after each iteration. The algorithm terminates when a desired level of tolerance is reached.

The summary of the iterative solving algorithm is graphically illustrated in Figure 1.

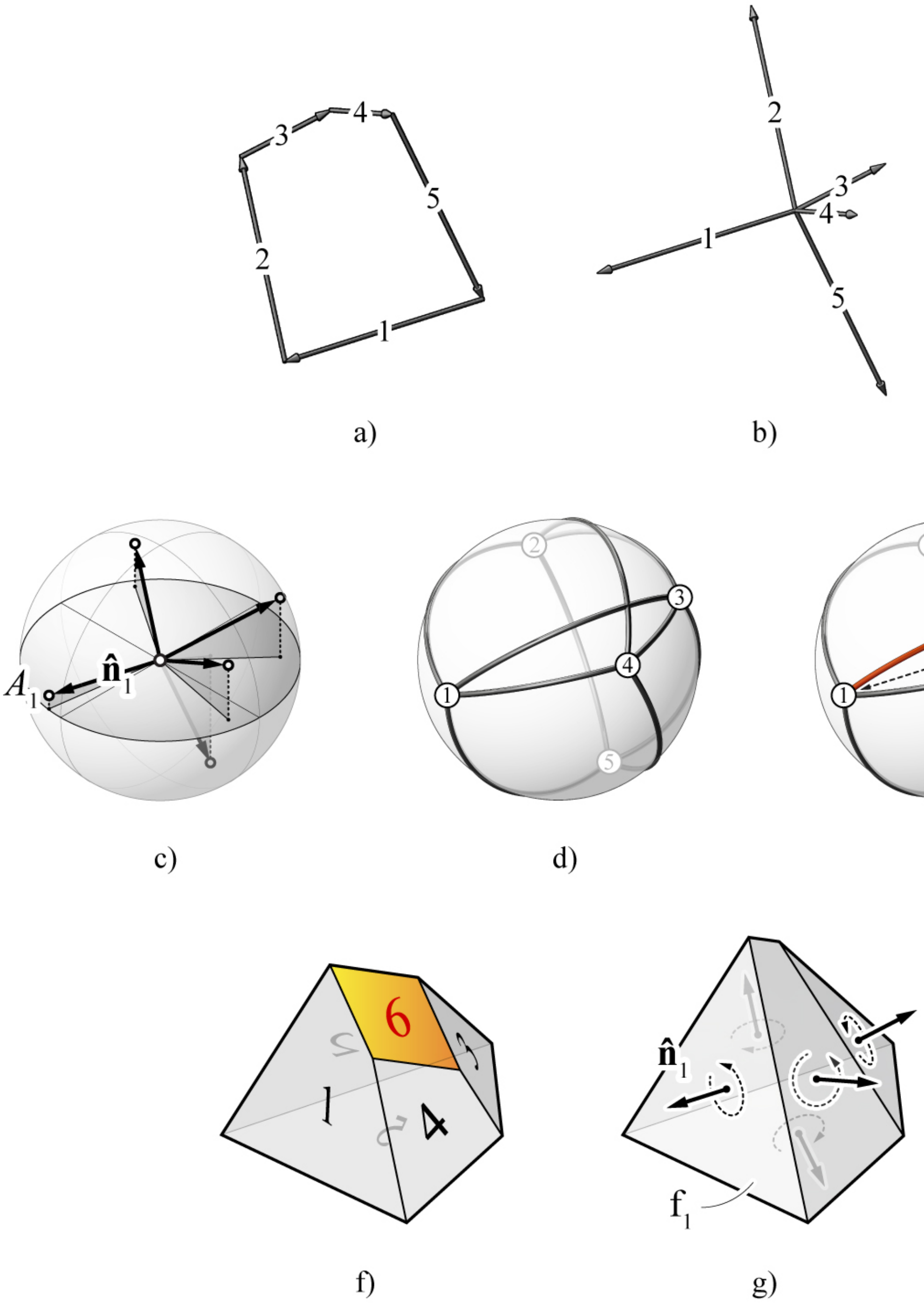


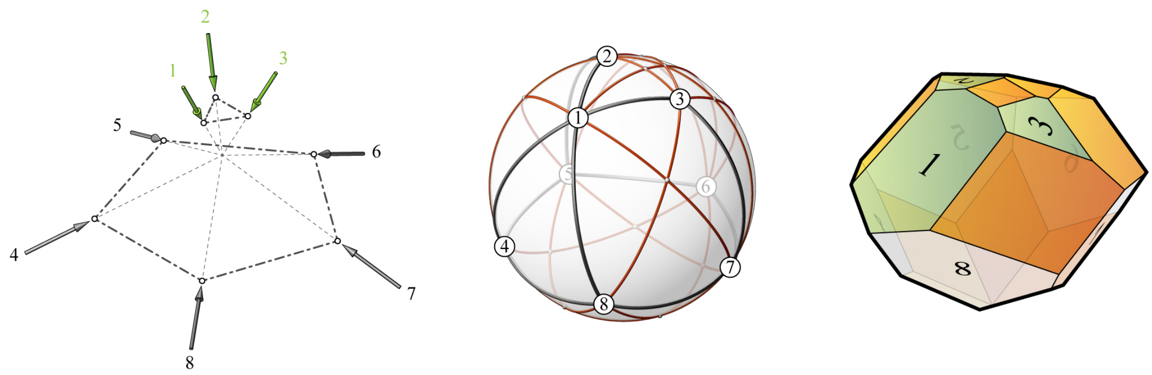
Figure 1: Polyhedral reconstruction procedure from prescribed force magnitudes and directions (or polyhedral face areas and orientations, respectively): a) spatial forces in equilibrium; b) “spike” representation of the Extended Gaussian Image (EGI); c) spherical representation of the EGI; d) EGI with arcs representing the face adjacencies; e) EGI with inclusion of detected cross-adjacencies; f) the unit polyhedron with the zero-area face highlighted in orange ; and g) the final polyhedral geometry.

3. Example

This section presents two types of examples for a simple loading scenario, where various boundary condition constraints result in different geometries of the GFP. The first set of examples will use fixed face orientations, and the second set of examples will use fixed prescribed areas as input constraints.

3.1. The setup

Consider the loading scenario in Figure 2. There are three applied point loads, and five pinned supports, which thus represents a static indeterminacy of degree two. The magnitudes, orientations and the locations of the applied loads (forces 1, 2 and 3) are fixed and are not allowed to change in any of the examples. In order to easily compare the changing locations of the supports, the supports are coplanar in all of the examples (i.e. a shell structure with its supports on a flat surface), and their positions can be either fixed or unconstrained on that plane. Each applied load will represent one unit force, and the magnitudes of the forces at the supports are assigned and measured proportionally to this unit force. For each example, a table showing which constraints were used as input is shown. An empty slot in the table means no constraint was assigned. In the form diagram, for each example, force vectors are shown with dots at 0.5 increments for visual reference.



a) b) c)

Figure 2: a) The loading scenario, with three non-vertical point loads and five pin supports (the hidden lines are shown for clarity and legibility only, and do not necessarily imply any specific structural topology); b) the EGI of the loading scenario; and c) the unit polyhedron with zero-area faces shown in orange, and the applied load faces shown in green.

3.2. Fixed face orientations

In the first set of examples, all face orientations are constrained to remain the same. In addition to the orientation constraints, assigning specific target areas for all faces will potentially result in an over-constrained scenario where the forces may not sum to zero and therefore a polyhedron may not exist that satisfies the input constraints. Therefore, only some of the faces are given specific target areas. Because the face orientations are constrained, only the magnitudes of the forces at the supports can change; the support locations do not change in these examples.

In the example in Figure 3a, all faces corresponding to the support reaction forces were given a target area of 1. However, the converged solution does not result in a polyhedron with the correct face areas, suggesting that indeed the inputs caused the problem to be over-constrained, and either the target area or the orientation criteria for some of the faces need to be unconstrained. In Figure 3b and c, fewer faces are given prescribed areas, and the solution converges much more accurately as expected.

3.3. Prescribed target areas

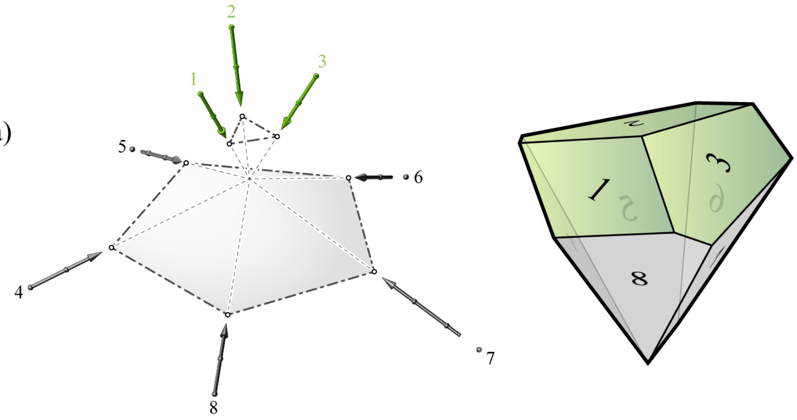
In some structural design applications, the magnitude of the reaction force at a support may be more important than its location or orientation. In the second set of examples in Figure 4, the orientation of the reaction forces at the supports are allowed to change freely. Unlike the examples in Figure 3, all support faces can be assigned target areas, and the correct orientations of the faces will be found that satisfies the target area constraints. In these examples, the resulting polyhedron satisfies the target area constraints, but the supports are relocated to new positions.

4. Conclusion

This paper illustrated how the boundary condition constraints can be addressed during the construction of the GFP. With a GFP that satisfies force-based constraints and criteria, it can be reintegrated with previously demonstrated form-finding explorations using polyhedral reciprocal diagrams to address more realistic structural design problems. The examples presented in this paper demonstrates that applications of 3D graphic statics can be extended beyond mere shape explorations to address more force-driven boundary condition constraints and more quantitative and practical structural design criteria.

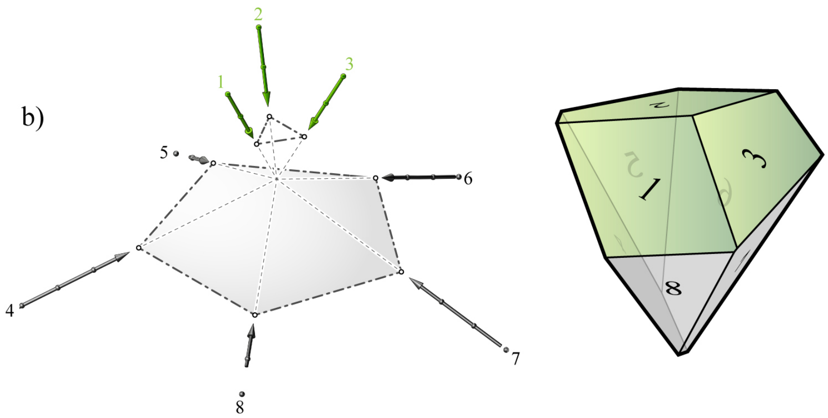
Topics for future work may include: addressing parallel loads or resolving multiple coplanar faces; extending the algorithm to reconstructions of concave polyhedra, which represents a GFP for a structure where both pinned and roller supports can be present; and improving the computational efficiency of the algorithm for cases with a disproportionally large number of zero-area faces.

a)



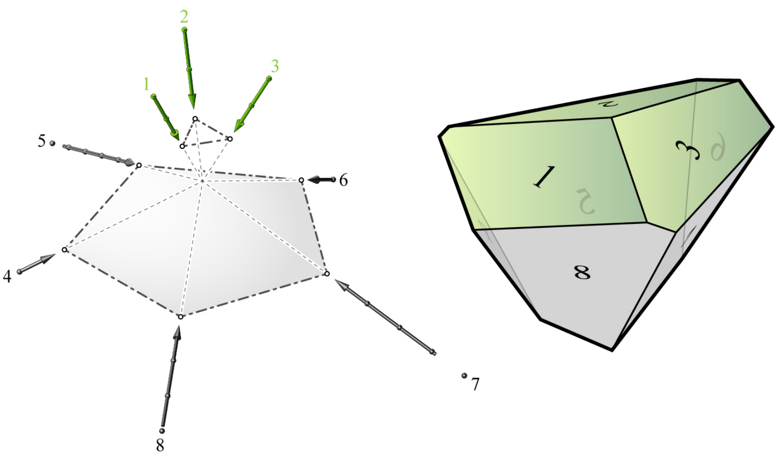
	Imposed		Result
	Orientation	Area	Area
1	✓	1.0	1.000
2	✓	1.0	0.999
3	✓	1.0	0.999
4	✓	1.0	0.998
5	✓	1.0	1.124
6	✓	1.0	0.698
7	✓	1.0	1.124
8	✓	1.0	0.998

b)



	Imposed		Result
	Orientation	Area	Area
1	✓	1.0	1.010
2	✓	1.0	1.012
3	✓	1.0	0.978
4	✓	-	1.799
5	✓	0.5	0.543
6	✓	1.5	1.470
7	✓	-	1.453
8	✓	0.5	0.529

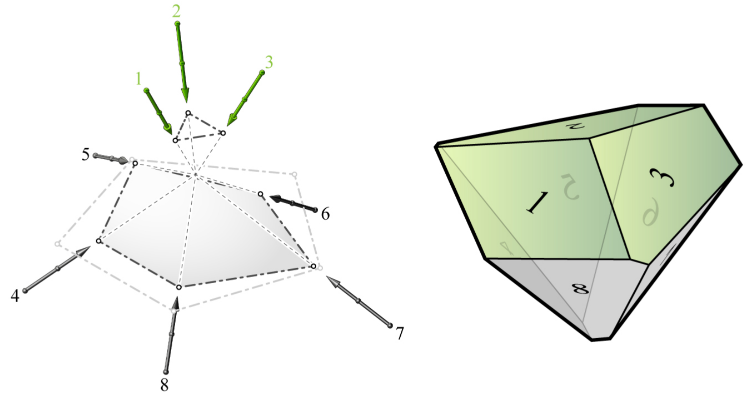
c)



	Imposed		Result
	Orientation	Area	Area
1	✓	1.0	1.022
2	✓	1.0	0.989
3	✓	1.0	1.011
4	✓	0.5	0.500
5	✓	-	2.157
6	✓	0.5	0.486
7	✓	-	1.563
8	✓	-	1.419

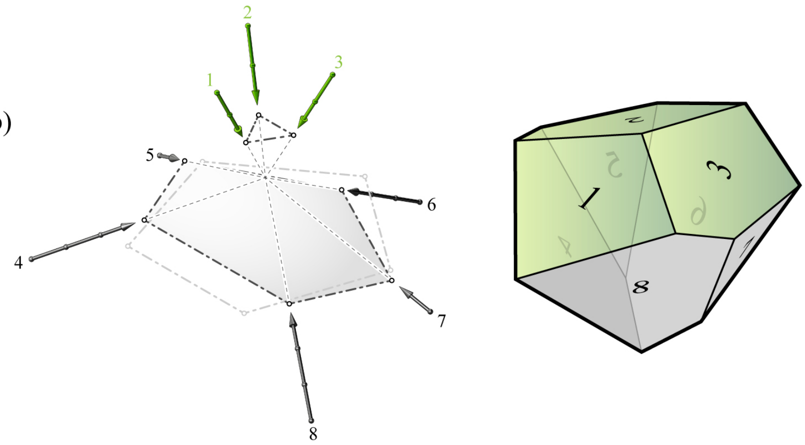
Figure 3: Examples where the orientation of the support faces are not allowed to change.

a)



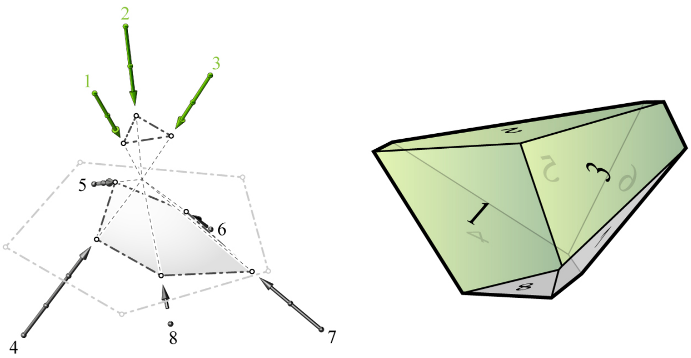
	Imposed		Result
	Orientation	Area	Area
1	✓	1.0	1.000
2	✓	1.0	0.999
3	✓	1.0	0.999
4	-	1.0	1.000
5	-	1.0	1.000
6	-	1.0	1.000
7	-	1.0	1.000
8	-	1.0	1.000

b)



	Imposed		Result
	Orientation	Area	Area
1	✓	1.0	4.999
2	✓	1.0	4.999
3	✓	1.0	4.999
4	-	1.5	1.500
5	-	0.5	0.500
6	-	1.5	1.499
7	-	0.5	0.500
8	-	1.5	1.500

c)



	Imposed		Result
	Orientation	Area	Area
1	✓	1.0	0.999
2	✓	1.0	0.999
3	✓	1.0	1.000
4	-	1.5	1.501
5	-	0.8	0.800
6	-	0.4	0.400
7	-	1.0	1.000
8	-	0.3	0.300

Figure 4: Examples where the orientations of the support faces are free to change, and target areas can be assigned for all faces.

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