Mechanics Research Communications 107 (2020) 103557

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

# Piecewise rigid displacement (PRD) method: a limit analysis-based approach to detect mechanisms and internal forces through two dual energy criteria

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ARTICLE INFO

*Article history:*
Received 26 February 2020
Accepted 25 June 2020
Available online 29 June 2020

*Keywords:*
Masonry
Limit analysis
Unilateral contact
Complementary energy
Settlements
Rigid no-tension material

ABSTRACT

This paper proposes an extension of the Piecewise Rigid Displacement (PRD) method based on a new dual linear programming problem that minimises the complementary energy. Before, the PRD method had been applied to solve the kinematical problem for masonry-like structures composed of normal, rigid, no-tension (NRNT) material minimising the total potential energy. Specifically, the PRD method frames this minimum-energy search as a linear programming problem whose solutions are displacements and singular strain fields (cracks).

Here, we show that the corresponding dual linear programming problem discretises the minimum of the complementary energy and returns, as solutions, admissible internal stress states compatible with the crack pattern obtained by solving the primal problem. Thus, these two minimum-energy criteria are dually connected, and their combined use allows coupling mechanisms and internal forces with settlements or homogeneous boundary displacements. This allows addressing different mechanical problems: equilibrium and stability of the reference configuration, effects of settlements, and mechanisms due to overloading (e.g. horizontal forces). Since the NRNT material represents the extension to continuum media of Heyman's material model, the PRD method offers an extremely fast, limit analysis-based, displacement approach that allows simultaneously finding mechanisms and compatible internal forces for any boundary condition, loads and geometry.



## 1. Introduction

Stability of unreinforced masonry (URM) structures relies primarily on their geometry rather than on material strength [1]. Because of the negligible capacity in tension of masonry, common sophisticated structural analysis methods, successfully employed for other construction and primarily based on the material failure, cannot be applied, mainly because of their inability to catch zero-energy modes [2,3], which are very common in masonry structures [4]. Indeed, because of the constitutive unilateral behaviour of masonry, whenever a construction is exposed to small changes in the external environment (e.g. settlements), it cracks, decomposes into rigid macroblocks and accommodates those changes in a new stable configuration [5]. A good model cannot ignore this peculiar behaviour. Several works that model the material with unilateral constitutive relations have been proposed [6–8]. In the current paper, neglecting the elastic deformation in compression, we assume as constitutive relation the normal, rigid, no-tension model (NRNT) [9], which frames the Heyman material relations [10] into continuum mechanics and, more importantly, allows for the application of Limit Analysis. The Heyman model, even though is a crude idealisation of the masonry behaviour, is commonly considered as one of the best approaches to look for the ultimate state of masonry structures exposed to overloads or foundation displacements. Despite that some robust numerical methods have been proposed to computationally solve the equilibrium problem [11–15,37], the solution of the kinematical problem is usually reached making simplified assumptions on the shape of the mechanism. Moreover, while the former can give only in some cases a rough idea of the potential crack pattern, the latter approaches are restricted to very simple geometries for which the mechanism is qualitatively known in advance. In any case, there still is the need for a general, displacement-based approach. For this purpose, the piecewise rigid displacement (PRD) method was conceived, solving the kinematical problem through the minimisation of the potential energy [16–18]. In the present paper, we show how the original minimum energy search gives rise to a new, dual linear

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https://doi.org/10.1016/j.mechrescom.2020.103557

programming problem, which represents the minimum of the complementary energy for NRNT materials [19]. We show that, by solving them simultaneously, the PRD method allows coupling mechanisms/cracks with internal stress states. The PRD method can be applied to any geometry and can address several mechanical scenarios: stability of the initial configuration, effects of overloading and, most importantly, modelling foundation settlements. In particular, the latter represents the main challenge of this research field since the settlements are the main cause of the common crack patterns that can be seen in masonry structures [1,4,5,20,34].

## 2. The Heyman model framed into continuum mechanics

In what follows, a 2D masonry structure is modelled as a continuum occupying the region $\Omega$. Let $\mathbf{T}$ be the stress in $\Omega$, $\mathbf{u}$ the displacement field and $\mathbf{E}$ the corresponding infinitesimal strain field. Let the boundary $\partial\Omega$, having $\mathbf{n}$ as unit outward normal, be partitioned into the constrained part $\partial\Omega_{\mathrm{D}}$, where the displacement field assumes the value $\bar{\mathbf{u}}$, and the loaded part $\partial\Omega_{\mathrm{N}}$ where the applied load is $\bar{\mathbf{s}}$.

### 2.1. NRNT material

The Heyman material model can be framed into continuum mechanics as a material characterised by these relations:

$$\mathbf{T} \in \mathrm{Sym}^{-}, \ \ \mathbf{E} \in \mathrm{Sym}^{+}, \ \ \mathbf{T} \cdot \mathbf{E} = 0, \tag{1}$$

where $\mathrm{Sym}^{-}$and $\mathrm{Sym}^{+}$are the mutual polar cones of semidefinite negative and positive symmetric tensors, respectively. Relations (1) define the normal, rigid, no-tension (NRNT) material [9] and, are the necessary ingredients to apply limit analysis since they are equivalent to the normality and dual-normality conditions [9].

### 2.2. Boundary value problem

The Boundary Value Problem (BVP) for a continuum composed of NRNT material can be summarised as:

$$\mathbf{E} = \mathrm{Sym}\nabla\mathbf{u}, \ \ \mathbf{E} \in \mathrm{Sym}^{+}, \ \ \mathbf{u} = \bar{\mathbf{u}} \text{ on } \partial\Omega_{\mathrm{D}}, \tag{2}$$

$$\mathrm{div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \ \ \mathbf{T} \in \mathrm{Sym}^{-}, \ \ \mathbf{Tn} = \bar{\mathbf{s}} \text{ on} \partial\Omega_{\mathrm{N}}, \tag{3}$$

$$\mathbf{T} \cdot \mathbf{E} = 0. \tag{4}$$

where $\mathrm{Sym}\nabla\mathbf{u}$: $=(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})/2$. A solution of the BVP is a triplet ($\mathbf{u}^{*}$, $\mathbf{E}^{*}$, $\mathbf{T}^{*}$) satisfying Eqs. (2–4) simultaneously. In what follows, we extensively use the sets of admissible displacement $\mathcal{K}$ and stress $\mathcal{H}$ fields:

$$\mathcal{K} = \left\{ \mathbf{u} \in \mathrm{SBV} \ / \ \mathrm{Sym}\nabla\mathbf{u} \in \mathrm{Sym}^{+} \ \& \ \mathbf{u} = \bar{\mathbf{u}} \text{ on } \partial\Omega_{\mathrm{D}} \right\}, \tag{5}$$

$$\mathscr{H} = \left\{ \mathbf{T} \in \mathrm{SBM} \ / \ \mathrm{div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \ \ \mathbf{T} \in \mathrm{Sym}^{-}, \ \ \mathbf{Tn} = \bar{\mathbf{s}} \text{ on} \partial\Omega_{\mathrm{N}} \ \right\}, \tag{6}$$

in which SBV and SBM are the set of special bounded variation functions and the set of special bounded measures [21], respectively. Thus, a triplet ($\mathbf{u}^{*}$, $\mathbf{E}^{*}$, $\mathbf{T}^{*}$) is a solution of the BVP if and only if $\mathbf{u}^{*} \in \mathcal{K}$, $\mathbf{T}^{*} \in \mathcal{H}$, and $\mathbf{T}^{*} \bullet \mathbf{E}^{*} = 0$.

**Remark 1.** Note that for this formulation the domain $\Omega$ has to be closed on $\partial\Omega_D$ [17], while the emerging stress tensor $\mathbf{s}(\mathbf{T})$ on the boundary (i.e. the trace of $\mathbf{T}$) can be written in the Cauchy form $\mathbf{Tn}$ only if the stress $\mathbf{T}$ is regular, otherwise, if the stress $\mathbf{T}$ crossing the boundary in $\mathbf{X} \in \partial\Omega$ is singular (i.e. a uniaxial stress in the form of a line Dirac delta distribution), the trace has to be written as $\mathbf{s}(\mathbf{T}) = P\,\delta(\mathbf{X})\,\mathbf{t}$ where the scalar $P$ is the magnitude of the singular stress and $\mathbf{t}$ is a unit vector directed as the uniaxial stress $\mathbf{T}$ [14,17]. ■

### 2.3. The kinematical and the equilibrium problems

Following the approach used in [9,19], every time we are looking for an element of $\mathcal{K}$ solving the BVP, we are solving the *kinematical problem* (KP); conversely, searching for an element of $\mathcal{H}$ solving the BVP means solving the *equilibrium problem* (EP) [30,35,36]. KP and EP are defined as compatible if $\mathcal{K}$ and $\mathcal{H}$ are not void, respectively. Two approaches can be adopted to solve KP and EP:

- a *displacement approach*, that is, by looking for a displacement field $\mathbf{u} \in \mathcal{K}$ for which there exists a stress field $\mathbf{T} \in \mathcal{H}$ such that $\mathbf{T} \bullet \mathbf{E}(\mathbf{u}) = 0$;
- an *equilibrium approach*, that is, by looking for a stress field $\mathbf{T} \in \mathcal{H}$ for which there exists a displacement field $\mathbf{u} \in \mathcal{K}$ such that $\mathbf{T} \bullet \mathbf{E}(\mathbf{u}) = 0$.

We will show that the PRD method gives rise to two dual linear programming (LP) problems: the primal solves the KP using a displacement approach, while its dual LP problem solves the EP with an equilibrium approach.

### 2.4. Singular stress and strain fields

To account for singularities involving both displacements $\mathbf{u}$ and stresses $\mathbf{T}$ [22,23], the strain $\mathbf{E}$ and the stress $\mathbf{T}$ can be additively decomposed into regular $(\bullet)^{\mathrm{r}}$ and singular $(\bullet)^{\mathrm{s}}$ parts:

$$\mathbf{E} = \mathbf{E}^{\mathrm{r}} + \mathbf{E}^{\mathrm{s}}, \ \ \mathbf{T} = \mathbf{T}^{\mathrm{r}} + \mathbf{T}^{\mathrm{s}} \ . \tag{7}$$

From here on, we neglect the regular part $\mathbf{E}^{\mathrm{r}}$ of the latent strain $\mathbf{E}$. Even though it is possible to account for the regular part $\mathbf{E}^{\mathrm{r}}$ [16,20], the reason behind the use of only singular strains arises directly from the observation of the peculiar behaviour of URM structures that when severely shaken by an earthquake or subjected to settlements, decompose into rigid macro-blocks [1,5,10,24] identified by a finite number of cracks. Mathematically, these cracks can be modelled as lines where displacement jumps $[\mathbf{u}]$ occur [17]. The best approach to model these jumps is to use Dirac delta functions. Therefore, let the curve $\Gamma$ be the support of a discontinuity (crack) and $\mathbf{n}$ and $\mathbf{t}$ the normal and tangential unit vectors to $\Gamma$, respectively. The strain is singular and can be written as

$$\mathbf{E} = v\ \delta(\Gamma)\ \mathbf{n} \otimes \mathbf{n} + \frac{1}{2} w\ \delta(\Gamma)\ (\mathbf{t} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{t}), \tag{8}$$

in which $\delta(\Gamma)$ is the Dirac delta function having $\Gamma$ as support and v and w are:

$$v = [\mathbf{u}] \cdot \mathbf{n}, \ \ w = [\mathbf{u}] \cdot \mathbf{t} \ . \tag{9}$$

Moreover, the material assumption $\mathbf{E} \in Sym^{+}$ restricts Eqns. (9) to be such that [16]

$$v \ = [\mathbf{u}] \cdot \mathbf{n} \ \geq 0, \ \ w = [\mathbf{u}] \cdot \mathbf{t} \ = 0 \ . \tag{10}$$

Therefore, for NRNT materials, allowing singular strain fields only, it results that

$$\mathbf{E} = v\ \delta(\Gamma)\ \mathbf{n} \otimes \mathbf{n} \ . \tag{11}$$

Similarly, the singular part of the stress field, having a curve $\Gamma$ as support, can be written as

$$\mathbf{T}^{\mathrm{s}} = P\ \delta(\Gamma)\ \mathbf{t} \otimes \mathbf{t}, \tag{12}$$

where $\mathbf{t}$ is the unit tangent vector to the curve $\Gamma$ and the scalar $P$ is the magnitude of the concentrated stress. The material restriction $\mathbf{T} \in \mathrm{Sym}^{-}$ implies $P \leq 0$ [22].

## 3. The search for a solution through two energy criteria

In this section, we recall the two energy criteria that can be used to solve the BVP for NRNT materials even in the presence of singular fields . The first one concerns the minimisation of the total potential energy $\wp(\mathbf{u})$ while the second one regards the minimisation of the complementary energy $\wp_c(\mathbf{T})$.

With the first criterion, following a displacement approach, we look for a minimiser of $\wp(\mathbf{u})$ in the set of kinematically admissible displacements $\mathcal{K}$; conversely, the second criterion, following an equilibrium approach, consists in the search for a minimum of $\wp_c(\mathbf{T})$ in the space $\mathcal{H}$ of admissible stress states.

### 3.1. Minimum of the total potential energy

Since the NRNT model takes into account neither the elastic nor the interface energy, the total potential energy $\wp(\mathbf{u})$ reduces to its linear part

$$\wp(\mathbf{u}) = -\int_{\partial\Omega_N} \bar{\mathbf{s}} \cdot \mathbf{u}\ \mathrm{ds} - \int_{\Omega} \mathbf{b}\cdot\mathbf{u}\ \mathrm{da}\ . \tag{13}$$

As proved in [17], the solution of the BVP is the minimiser $\mathbf{u}^*$ of $\wp(\mathbf{u})$, that is, the displacement field $\mathbf{u}^*$ such that:

$$\wp(\mathbf{u}^*) = \min_{\mathbf{u}\in\mathcal{K}} \wp(\mathbf{u})\ . \tag{14}$$

### 3.2. Minimum of the complementary energy

For NRNT materials also the complementary energy loses the nonlinear part and reduces to

$$\wp_c(\mathbf{T}) = -\int_{\partial\Omega_D} \mathbf{s}(\mathbf{T}) \cdot \bar{\mathbf{u}}\ \mathrm{ds}, \tag{15}$$

in which $\mathbf{s}(\mathbf{T})$ is the trace of the stress tensor on the constrained boundary $\partial\Omega_D$. As shown in [19], when using an equilibrium approach, the search for a solution of the BVP can be carried out finding a minimiser $\mathbf{T}^*$ of the complementary energy:

$$\wp_c(\mathbf{T}^*) = \min_{\mathbf{T}\in\mathcal{H}} \wp_c(\mathbf{T})\ . \tag{16}$$

**Remark 2.** It is worth to recall the result on the minimum principle (13) shown in [17]: if the EP is compatible ($\mathcal{H} \neq \emptyset$) and the KP is homogeneous, $\mathbf{u}^* = \mathbf{0}$ is a minimum solution. Furthermore, any $\mathbf{T} \in \mathcal{H}$ constitutes an admissible stress solution since $\mathbf{T} \bullet \mathbf{E}(\mathbf{u}^*) = 0$, $\forall\ \mathbf{T} \in \mathcal{H}$. The physical meaning of this results is the following: when a structure is not subjected to foundation displacements (i.e. homogeneous boundary condition), and an admissible stress state exists ($\mathcal{H} \neq \emptyset$), the structure is stable in its configuration, no cracks occur (i.e. $\mathbf{u}^* = \mathbf{0}$ is a solution) and any element of $\mathcal{H}$ is an admissible stress solution. This result perfectly fits the spirit of Limit Analysis [10], since if there are no settlements, and the load is compatible, there are infinite possible admissible stress states unless $\mathcal{H}$ is a singleton (i.e. the structure is in a limit state). ■

## 4. PRD method: two dual energy-based LP problems

In this section, after recalling how the PRD method reduces the minimum problem (14) to a linear programming (LP) problem, we show how its dual LP problem represents the discretization of the minimum problem (16). The combined use of these two dual energy-criteria allow coupling mechanisms, that are solutions of the primal problem, with internal and external forces, solving the corresponding dual problem.

### 4.1. The primal problem: minimum of the total potential energy

For each polygonal, M-element partition of $\Omega$

$$(\Omega_i)_{i\in\{1,\ \dots,\ M\}}, \tag{17}$$

the PRD method, as shown in [17,18], allows to dicretise and transform the minimum-energy problem (14) into the following LP problem

$$\text{minimise}\quad -\mathbf{c}\cdot\mathbf{U} \tag{18}$$

$$\text{subject to}\quad \mathbf{A}^{\mathrm{int}}_{\mathrm{ub}}\mathbf{U} \geq \mathbf{0},\ \ \mathbf{A}^{\mathrm{int}}_{\mathrm{eq}}\mathbf{U} = \mathbf{0}, \tag{19}$$

$$\mathbf{A}^{\mathrm{ext}}_{\mathrm{ub}}\mathbf{U} \geq -\bar{\boldsymbol{\delta}}_n,\ \ \mathbf{A}^{\mathrm{ext}}_{\mathrm{eq}}\mathbf{U} = -\bar{\boldsymbol{\delta}}_t, \tag{20}$$

where the internal material restrictions (10) are enforced by matrix relations (19) and the non-homogeneous boundary conditions through (20), in which the vector $\bar{\boldsymbol{\delta}}_n$ [$\bar{\boldsymbol{\delta}}_t$] collects the normal [tangential] components of the prescribed boundary displacement. The objective function (18) is the total potential energy of the external loads, that is, the opposite of the work done by the external forces $\mathbf{c}$ (two forces and a torque per element) and the corresponding Lagrangian parameters $\mathbf{U}$ of the elements of the partition (17). With $\ell_i$ [$\ell_c$] the number of the internal [constrained (i.e. the ones lying on the constrained boundary)] interfaces (i.e. the ones common for two elements), both matrices in (19) [(20)] are composed of 3 M columns and $2\ell_i$ [$2\ell_c$] rows.

We will refer to the LP problem (18–20) as the PRD method's primal problem (P).

**Remark 3.** Relations (19, 20) define the subset $\mathbb{K}_{PRD}$ [17] of the space $\mathcal{K}$ of kinematically admissible displacements. $\mathbb{K}_{PRD}$ represents the space of all possible mechanisms generated by the original partition (17), and, thus, the energy criterion (19) selects, amongst all the $\infty^{3M}$ mechanisms belonging to $\mathbb{K}_{PRD}$, the one/ones solving the BVP. ■

### 4.2. The dual problem: minimum of the complementary energy

The LP problem (P) can be written as the equivalent problem (P'):

$$\text{maximise}\quad \mathbf{c}\cdot\mathbf{U} \tag{21}$$

$$\text{subject to}\quad \mathbf{A}_{\mathrm{ub}}\mathbf{U} \geq -\boldsymbol{\delta}_n \tag{22}$$

$$\mathbf{A}_{\mathrm{eq}}\mathbf{U} = -\boldsymbol{\delta}_t, \tag{23}$$

where $\mathbf{A}_{\mathrm{ub}}$ [$\mathbf{A}_{\mathrm{eq}}$] collects vertically the matrices $\mathbf{A}^{\mathrm{int}}_{\mathrm{ub}}$ [$\mathbf{A}^{\mathrm{ext}}_{\mathrm{ub}}$] and $\mathbf{A}^{\mathrm{ext}}_{\mathrm{ub}}$ [$\mathbf{A}^{\mathrm{ext}}_{\mathrm{eq}}$] and $\boldsymbol{\delta}_n$ [$\boldsymbol{\delta}_t$] collects the zero vectors arising from the inequalities [equalities] and the potential non-zero vector $\bar{\boldsymbol{\delta}}_n$ [$\bar{\boldsymbol{\delta}}_t$] collecting the normal [tangential] components of the displacements on the constrained boundary. Now, consider this new linear programming problem (D):

$$\text{minimise}\quad -\mathbf{f}_n\cdot\boldsymbol{\delta}_n - \mathbf{f}_t\cdot\boldsymbol{\delta}_t \tag{24}$$

$$\text{subject to}\quad \mathbf{A}^{T}_{\mathrm{ub}}\,\mathbf{f}_n + \mathbf{A}^{T}_{\mathrm{eq}}\,\mathbf{f}_t = \mathbf{c} \tag{25}$$

$$\mathbf{f}_n \leq \mathbf{0},\ \ \mathbf{f}_t \in \mathbb{R}^{2(\ell_i+\ell_c)}\ . \tag{26}$$

The LP problem (D) is the linear program dual to the linear program (P'). Furthermore, since (P) and (P') are equivalent, the LP problem (D) is also the dual of the LP problem (P). From here on, we call the problem (D) as the dual problem and the *original* problem (P) the primal problem.

In particular, $\mathbf{f}_\mathrm{n}$ and $\mathbf{f}_\mathrm{t}$ are both $2(\ell_\mathrm{i}+\ell_\mathrm{c})$ dimensional vectors, the former representing the normal forces dual to the unilateral constraints (22), while the latter the tangential forces dual to the bilateral (equality) constraints (23). The matrix expression (25) exactly represents the equilibrium of the elements composing the partition (17). Indeed, it is governed by the transpose of the kinematic matrices of the problem (P). The objective function (24) is the opposite of the product of dual forces $\mathbf{f}_\mathrm{n}$ and $\mathbf{f}_\mathrm{t}$ and the displacements $\boldsymbol{\delta}_\mathrm{n}$ and $\boldsymbol{\delta}_\mathrm{t}$.

Particularly, since the displacement field can be non-zero only on the constrained boundary, the objective function (24) reduces to the opposite of the product of the reaction forces and the (potential) non-zero boundary displacements. Therefore, it represents the complementary energy for an NRNT material.

Defining $\mathbf{f} = [\mathbf{f}_\mathrm{n}, \mathbf{f}_\mathrm{t}]$, $\boldsymbol{\delta} = [\boldsymbol{\delta}_\mathrm{n}, \boldsymbol{\delta}_\mathrm{t}]$ and $\mathbf{A}^\mathrm{T} = [\mathbf{A}^\mathrm{T}_\mathrm{ub}, \mathbf{A}^\mathrm{T}_\mathrm{eq}]$, the dual problem (D) which discretises the minimum of the complementary energy (16) reads: “Find a minimiser $\mathbf{f}^*$ of the complementary energy $\wp_c(\mathbf{f})$ in the set $\mathbb{H}_\mathrm{PRD}$, that is:

$$\wp_c(\mathbf{f}^*) = \min_{\mathbf{f}\in\mathbb{H}_\mathrm{PRD}} \left(-\,\mathbf{f}\cdot\boldsymbol{\delta}\right)'', \tag{27}$$

where:

$$\mathbb{H}_\mathrm{PRD} = \left\{\mathbf{f}\in\mathbb{R}^{4(\ell_\mathrm{i}+\ell_\mathrm{c})} \;/\; \mathbf{A}^\mathrm{T}\mathbf{f} = \mathbf{c},\; \mathbf{f}_\mathrm{n} \le \mathbf{0},\; \mathbf{f}_\mathrm{t}\in\mathbb{R}^{2(\ell_\mathrm{i}+\ell_\mathrm{c})}\right\}. \tag{28}$$

The set $\mathbb{H}_\mathrm{PRD}$ represents the space of the emerging stresses (i.e. point Dirac delta distributions) at the boundary of each element of the partition (17). In this sense, the minimum problem (D) provides a set of forces on the boundaries defined by (17), thus, both internal and external forces pattern (i.e. the trace of the emerging stress on the boundary of each element). We use the acronym PRD for the set $\mathbb{H}_\mathrm{PRD}$ to emphasise that the dual LP problem (D) provides a discretisation of the minimum problem (16) using the same partition adopted for the minimum problem (14). Moreover, note that the forces solving the dual problem (D) are compatible with the mechanisms/cracks solving the primal problem (P). Indeed, we will show that if there is a crack along an interface, e.g. two blocks hinged at a point, by solving the dual problem (D), the resultant force on that interface is a compressive force passing precisely through the relative centre of rotation (i.e. the contact point).

**Remark 4.** The solutions to the dual problems allow not only evaluating internal and external forces for a structure undergoing foundation displacements; it provides something more. As shown in **Remark 2**, if the EP is compatible and the KP is homogeneous, $\mathbf{u}^* = \mathbf{0}$ is a minimum solution and, thus, any element $\mathbf{T}\in\mathcal{H}$ represents a solution of the equilibrium problem. This can be easily seen by looking at the dual problem (D). In the case of homogeneous boundary displacements, the objective function is constant and zero, and if the linear constraints defining the set $\mathbb{H}_\mathrm{PRD}$ are such that the set $\mathbb{H}_\mathrm{PRD}$is not void and not a singleton, infinite admissible solutions are possible. These solutions represent safe internal stress states in the sense of Limit Analysis. ■

**Remark 5.** If the energy criterion (14), selecting the mechanism solving the BVP, represents an application of the *Kinematic Theorem* of Limit Analysis, the energy criterion (16) (and its discretised form (24–26)) represents a selector of an equilibrated stress state and, thus, gives a way to apply the **Safe Theorem**. As observed in **Remark 4**, when the structure is not subjected to foundation settlements, infinite solutions are possible. Conversely, if the structure is subjected to foundation displacements, the solution of the primal problem (P) returns a rigid macro-block partition and matching cracks. Thus, by solving the dual problem (D), one can find an equilibrated, internal stress state compatible with that crack. This solution could be either globally or locally unique; that is, the solution is unique for the moving part of the structure activated by the foundation displacements since the moving part becomes statically determinate. Of course, the remaining part of the structure could still be statically indeterminate, and for that part, the solution is not unique. ■

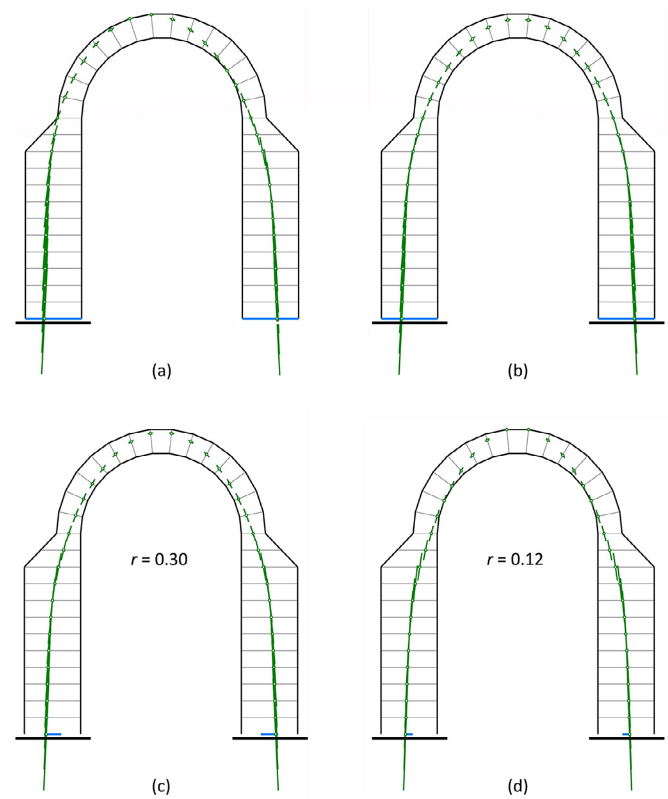


**Fig. 1.** The solution of the primal problem (P) is $\mathbf{u}^* = \mathbf{0}$, thus the structure is stable. In (a,b), two solutions of the dual problem (D): the structure is statically indeterminate and the set $\mathbb{H}_\mathrm{PRD}$ is composed of infinite admissible stress states $\mathbf{T}$. In (c,d), two admissible stress states obtained by constraining the flow of forces to cross a tighter support base, symmetrically reduced through a scale factor $r$.

**Remark 6.** A masonry structure accommodates small foundation displacements through a rigid macro-block partition defined by specific cracks. These fractures allow understanding how the structure behaves mechanically and, thus, defining (at least locally, see Remark 5) a unique, compatible and equilibrated internal stress state. In this sense, the foundation displacements select an admissible stress state. ■

## 5. Numerical applications: a buttressed arch

In this section, looking at the buttressed arch, we perform some PRD analyses, taking into account different mechanical scenarios, to show how the PRD method represents an efficient numerical approach for computationally applying Limit Analysis à la Heyman to continuous media. The PRD analysis has been conducted using *compas_prd*, a Python-based computational tool, which, in its current implementation, is designed for planar, 2.5D analysis, that is, the analysis is planar but non-uniform, symmetrical orthogonal depths can be considered. The PRD method can be used to assess any complex geometry, but we chose a simple buttressed arch to demonstrate the method's potential clearly, addressing different mechanical problems. Studies on buttressed

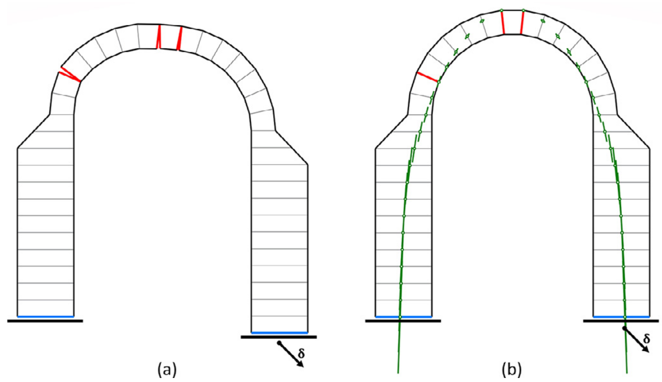


**Fig. 2.** Buttressed arch subjected to a foundation displacement. In (a), the solution of the primal problem (P) shows a three-hinge mechanism. In (b), the internal and external forces solving the corresponding dual problem (D) are represented by their resultants, labelled in green; the force resultants are perfectly compatible with the crack pattern (marked in red).

arches can be found in [31–33]. The geometry and its discretisation are depicted in Fig. 1. The semicircular arch has an internal radius of 1.00 m, a thickness of 0.30 m, a depth of 0.50 m and is discretised into 15 voussoirs. The two buttresses have a height of 2.5 m, a base of 0.70 m, a depth of 1.00 m and are partitioned into 12 elements having the same height. As load, the self-weight for a uniform distribution of mass density ($\rho = 1800\mathrm{kg/m^3}$) is considered. It is worth to point out that we will represent, in what follows, the solution of the dual problem (D) through the resultant forces acting on each interface. Each analysis presented has been conducted solving both LP problems with CVXPY [25] choosing MOSEK [26] as a solver, and the computational time required to solve both LP problems is about 0.04 s with an Intel® Core™i7–8850H.

**Remark 7.** Note that the solution of the primal problem (P) is a small displacement field (i.e. *small* compared to the overall size of the structure, see [17]). Therefore, in the following numerical applications, no information about the value of the prescribed displacement is provided. Indeed, if the prescribed displacements are not small, the structure can exhibit an evolution of the crack pattern, and, the unique way to account for this behaviour is to solve a sequence of LP problems on the updated geometry [27]. ■

### 5.1. Initial perfect configuration

The first analysis looks at the stability of the structure in the reference configuration, assuming homogeneous boundary displacements ($\overline{\mathbf{u}} = \mathbf{0}$). Solving the primal problem (P), we find that the structure is stable in its initial configuration since $\mathbf{u}^* = \mathbf{0}$ is the minimiser of the total potential energy. The objective function (24) of the corresponding dual problem is zero. As shown in **Remark 4**, if $\mathbb{H}_{\text{PRD}} \neq \emptyset$, any element of $\mathbb{H}_{\text{PRD}}$ is a solution of the equilibrium problem. Particularly, the constraints of the dual problem (D), represented by Eqns. (25,26), are such that $\mathbb{H}_{\text{PRD}}$ is not void. Indeed, when solving the corresponding dual problem (D) for homogeneous boundary displacements, infinite possible equilibrated solutions can be found. In Fig. 1**a and b**, two of them are represented. One of the possible approaches to explore other admissible stress solutions is to add more constraints, that is, e.g. enforcing the flow of forces to go through a reduced width at the base. In Fig. 1**c and d**, two admissible stress states obtained by reducing the actual base symmetrically with a scale factor $r$ are depicted. Specifically, $r = 0.12$ corresponds to the ultimate allowable value for which a compressive stress state lying within the structure can be still found. Finally, the PRD stability analysis in the reference configuration shows that no cracks form, the structure is stable but statically indeterminate since $\mathbb{H}_{\text{PRD}}$ is composed of infinite admissible stress solutions.

### 5.2. Foundation displacements

In this section, we perform a PRD analysis of the same structure subjected to a settlement of the base on the right. This foundation displacement, assumed to be uniformly distributed, is such that the ratio amongst its vertical and horizontal component is 3 (Fig. 2). The solution of the primal problem (P) returns a non-zero displacement field that minimises the total potential energy (Fig. 2**a**). The structure accommodates the new boundary condition decomposing into four rigid macro-blocks defined by three cracks. The interfaces affected by the crack pattern are labelled in red: they represent the locations of non-zero singular strain fields (i.e. displacement jumps). In Fig. 2**b**, the resultant of the internal and external forces, solving the corresponding dual problem (D), are depicted. They are compatible with the cracks obtained after solving the primal problem: when a hinge occurs, the corresponding force goes through the relative centre of rotation. Furthermore, the force resultants at all interfaces are everywhere within the structural geometry. Thus, the buttressed arch is safe according to the **Safe Theorem**. This is a consequence of the fact that the constraints of the dual problem (D) enforce these admissibility conditions directly: if the problem is feasible, only equilibrated compressive forces acting within the geometry are allowed.

### 5.3. Horizontal forces

The usual way to account for seismic actions is to simulate them with horizontal forces proportional to the self-weight using a scale factor $\lambda$. Once the scale factor $\lambda$ reaches a specific value, the structure becomes a mechanism [28]. Here, we show how the PRD method can be applied to also solve this problem. In Fig. 3, solutions of both the primal (P) and dual (D) problems for increasing values of $\lambda$ are reported. Notably, for a low value (Fig. 3**a**), the axis of symmetry of the flow of forces is almost vertical and the flow of forces does not touch the boundary. By increasing $\lambda$, the axis becomes more inclined (Fig. 3**b**). When $\lambda$ reaches the value of 0.22 (Fig. 3**c**), the buttressed arch becomes a four-hinge mechanism and the flow of forces touches the structural boundary in four points. Note that the base on the right (Fig. 3**c and d**) is labelled in red since it is affected by a displacement jump (the domain is close on the constrained boundary, see Remark 1). In this situation, the total potential energy is not bounded from below anymore and the structure becomes unstable [29] and starts oscillating (Remark 7). From this point on, the structure has to be analysed applying the laws of dynamics, e.g. applying the rocking model to the partition defined in Fig. 3**d**.

## 6. Conclusions

In this paper, we have shown how the original formulation of the PRD method, based on the minimum of the total potential energy [16–18], gives rise to a new dual linear programming problem based on the minimum of the complementary energy (the first proof and an application of this minimum energy criterion can be traced back to [19]). In this sense, these two minimum-energy criteria are dually connected. In particular, the dual problem allows the search for equilibrated singular internal stress states compatible with the displacement and crack patterns that are a solution of the original primal LP problem. Some typical mechanical analyses applied to a simple buttressed arch have been presented to show the main features and clear potential of the extended PRD method. Specifically, the new dual-energy criterion solves the

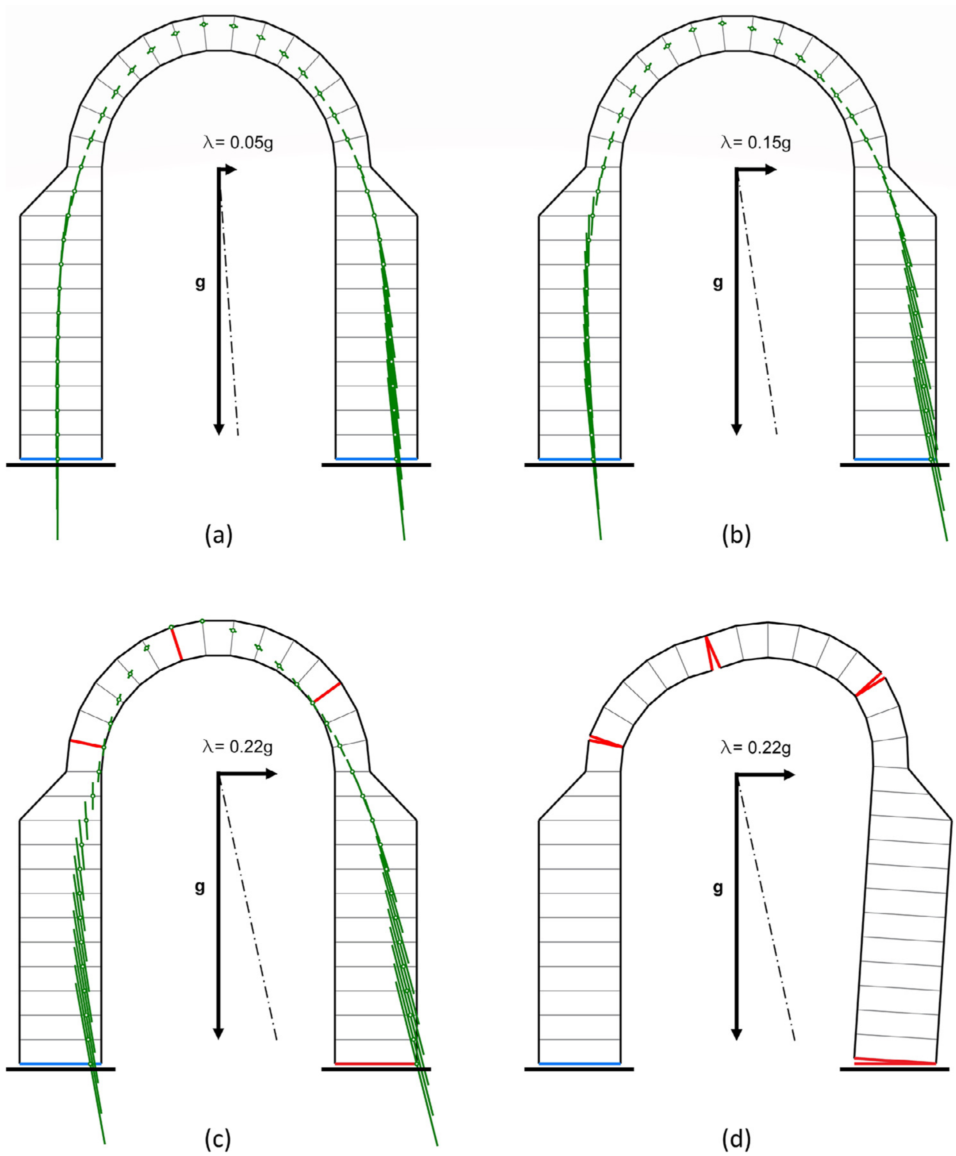


**Fig. 3.** The buttressed arch under horizontal forces (a-d). Until the horizontal static multiplier is less than 0.22, the structure is stable (a,b). When λ reaches the value 0.22, the buttressed arch becomes a four-hinge mechanism (c, d) and the corresponding resultants go through the relative centres of rotation defining the four hinges (c).

boundary value problem for either prescribed or homogeneous displacements. In the former case, the minimum of the complementary energy provides forces compatible with the mechanism, that is, with a solution of the primal problem; in the latter, if the structure is stable, it offers infinite admissible stress states unless the structure is in a limit-state condition (**Section 5.3**).

Since the constitutive relations defining the NRNT material are equivalent to the normality conditions, the PRD method represents an approach to (computationally) apply limit analysis à la Heyman to any planar masonry structure with no constraints on loads and geometry. Indeed, in Fig. 4, looking at a complex 2.5D model of an approximate cross-section of Amiens cathedral, the results, in terms of displacements and cracks (Fig. 4**a**), rigid macro-block partition (Fig. 4**b**) and internal forces (Fig. 4**c**), for a prescribed small foundation displacement are depicted. The total computational time required in this case to solve both dual linear programming problems is only 0.1 s. For further details, the reader is referred to [27], in which referring to this geometry, a displacement capacity analysis, showing the evolution of both mechanism and internal stress state, is performed. Finally, some further features of the PRD method need to be outlined. The analysis involves the whole structure, allowing the understanding in terms of mechanisms and forces of the global mechanical behaviour either under settlements or under general load conditions. For statically indeterminate structures, it provides infinite admissible stress solutions, which can be selected by a suitable criterion (**Section 5.1**); a prescribed settlement can be thought of as one of these criteria (**Section 5.2**). This is a peculiarity of masonry structures as pointed out by Heyman in several of his works. Finally, we have to point out that the PRD method is a displacement-based approach and, in this sense, it offers a direct technique to solve problems stemming from the real world where the displacements-like boundary conditions play a crucial role. This approach reverses the common approach followed by other methods, based on an equilibrium approach, which provide displacements because of internal equilibrium states (which can be infinite in a perfect original condition). Indeed, a hinge can form if and only if the corresponding displacement is compatible: the requirement for which the resultant goes

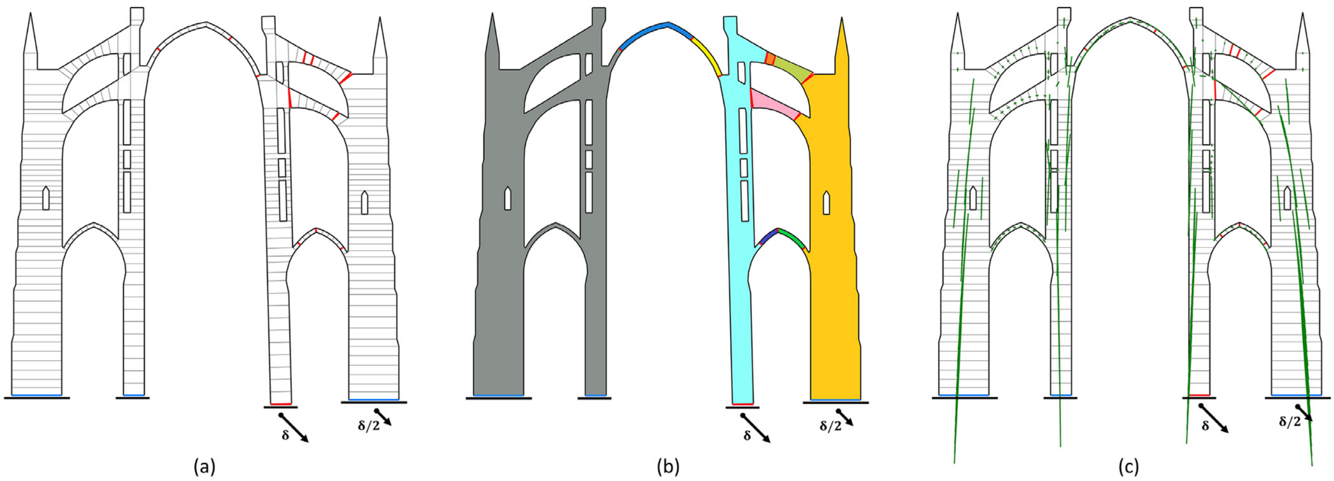


**Fig. 4.** PRD analysis of a more complex 2.5D model of an approximate cross-section of Amiens cathedral [27] subjected to settlements whose vertical inclination is 45°: (a) effects in terms of crack pattern (marked in red), (b) rigid macro-block partition of the domain and (c) internal forces (labelled in green) in equilibrium with the external loads and compatible with the crack pattern.

through the hinge is a necessary but not a sufficient condition for causing a crack [27].

## Funding

This work was supported by the SNSF - Swiss National Science Foundation [grant number 178953].

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## References

[1] J. Heyman, The structural engineer's view of ancient buildings, J. Mech. Mater. Struct. 13 (5) (2019) 609–615.
[2] P. Block, T. Ciblac, J.A. Ochsendorf, Real-time limit analysis of vaulted masonry buildings, Comput. Struct. 84 (29–30) (2006) 1841–1852.
[3] H.V. Shin, C.F. Porst, E. Vouga, J.A. Ochsendorf, F. Durand, Reconciling elastic and equilibrium methods for static analysis, ACM Transactions on Graphics (TOG) 35 (2) (2016) 1–16.
[4] J.A. Ochsendorf, Collapse of masonry structures, University of Cambridge, 2002.
[5] M. Como, Statics of Historic Masonry Constructions, Springer, Berlin, 2013.
[6] G. Romano, M. Romano, Sulla soluzione di problemi strutturali in presenza di legami costitutivi unilaterali, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 67 (1–2) (1979) 104–113.
[7] M. Giaquinta, E. Giusti, Researches on the equilibrium of masonry structures, Arch. Ration. Mech. An. 88 (4) (1985) 359–392.
[8] G. Del Piero, Constitutive equation and compatibility of the external loads for linear elastic masonry-like materials, Meccanica 24 (3) (1989) 150–162.
[9] M. Angelillo, Practical applications of unilateral models to Masonry Equilibrium, in: M. Angelillo (Ed.), Mechanics of Masonry Structures, Springer, Vienna, 2014, pp. 109–210.
[10] J. Heyman, The stone skeleton, Int. J. Solids Struct. 2 (2) (1966) 249–279.
[11] F. Fraternali, M. Angelillo, A. Fortunato, A lumped stress method for plane elastic problems and the discrete-continuum approximation, Int. J. Solids Struct. 39 (25) (2002) 6211–6240.
[12] P. Block, J. Ochsendorf, Thrust network analysis: a new methodology for three--dimensional equilibrium, J. IASS 48 (3) (2007) 167–173.
[13] F. Fraternali, A thrust network approach to the equilibrium problem of unreinforced masonry vaults via polyhedral stress functions, Mech. Res. Commun. 37 (2) (2010) 198–204.
[14] M. Angelillo, E. Babilio, A. Fortunato, Singular stress fields for masonry-like vaults, Continuum Mech. Therm. 25 (2–4) (2013) 423–441.
[15] P. Block, L. Lachauer, Three-dimensional funicular analysis of masonry vaults, Mech. Res. Commun. 56 (2014) 53–60.
[16] A. Iannuzzo, A new rigid block model for masonry structures, Università degli Studi di Napoli Federico II, 2017.
[17] A. Iannuzzo, M. Angelillo, E. De Chiara, F. De Guglielmo, F. De Serio, F. Ribera, A. Gesualdo, Modelling the cracks produced by settlements in masonry structures, Meccanica 53 (7) (2018) 1857–1873.
[18] A. Iannuzzo, Energy based fracture identification in masonry structures: the case study of the church of "Pietà dei Turchini", J, Mech. Mater. Struct. 14 (5) (2019) 683–702.
[19] M. Angelillo, A. Fortunato, A. Gesualdo, A. Iannuzzo, G. Zuccaro, Rigid block models for masonry structures, IJMRI 3 (4) (2018) 349–368.
[20] A. Iannuzzo, F. De Serio, A. Gesualdo, G. Zuccaro, A. Fortunato, M. Angelillo, Crack patterns identification in masonry structures with a C° displacement energy method, IJMRI 3 (3) (2018) 295–323.
[21] L. Ambrosio, M. Miranda Jr, D. Pallara, Special functions of bounded variation in doubling metric measure spaces, Calc. Var.: Top. Math. Herit. E. De Giorgi 14 (2004) 1–45.
[22] M. Angelillo, A. Fortunato, A. Montanino, M. Lippiello, Singular stress fields in masonry structures: derand was right, Meccanica 49 (5) (2014) 1243–1262.
[23] M. Lucchesi, M. Šilhavý, N. Zani, A new class of equilibrated stress fields for no-tension bodies, J. Mech. Mater. Struct. 1 (3) (2006) 503–539.
[24] S. Huerta Fernández, Geometry and equilibrium: the gothic theory of structural design, Struct. Eng. 84 (2) (2006) 23–28.
[25] S. Diamond, S. Boyd, CVXPY: a Python-embedded modeling language for convex optimization, J. Mach. Learn. Res. 17 (1) (2016) 2909–2913.
[26] A.P.S. Mosek, The MOSEK Optimization Software, Online Athttp://Www.mosek.com 54 (2–1) (2010).
[27] A. Iannuzzo, A. Dell'Endice, T. Van Mele, P. Block, Numerical Limit Analysis-based modelling of masonry structures subjected to large displacements, Comput. Struct. (2020) Submitted for publication.
[28] A. Iannuzzo, A. De Luca, A. Fortunato, A. Gesualdo, M. Angelillo, Fractures detection in masonry constructions under horizontal seismic forces, Ing. Sismica 35 (3) (2018) 87–103.
[29] A. Iannuzzo, C. Olivieri, A. Fortunato, Displacement capacity of masonry structures under horizontal actions via PRD method, J. Mech. Mater. Struct. 14 (5) (2019) 703–718.
[30] C. Cennamo, C. Cusano, M. Angelillo, On the statics of large domes: a static and kinematic approach for San Francesco di Paola in Naples, Proceedings of the International Masonry Society Conferences (2018) 504–517.
[31] G. Brandonisio, M. Angelillo, A. De Luca, Seismic capacity of buttressed masonry arches, Eng. Struct. 215 (2020).
[32] H. Alexakis, N. Makris, Hinging mechanisms of masonry single-nave barrel vaults subjected to lateral and gravity loads, J. Struct. Eng. 143 (6) (2017).
[33] P. Zampieri, M. Amoroso, C. Pellegrino, The masonry buttressed arch on spreading support, Structures 20 (2019) 226–236.
[34] K.E. Kurrer, *The history of the theory of structures: searching for equilibrium*, John Wiley & Sons, 2018.
[35] M. Monaco, I. Bergamasco, M. Betti, A no-tension analysis for a brick masonry vault with lunette, J. Mech. Mater. Struct. 13 (5) (2019) 703–714.
[36] A. Gesualdo, G. Brandonisio, A. De Luca, A. Iannuzzo, A. Montanino, C. Olivieri, Limit analysis of cloister vaults: the case study of Palazzo Caracciolo di Avellino, Journal of Mechanics of Materials and Structures 14 (5) (2019) 739–750.
[37] E. De Chiara, C. Cennamo, A. Gesualdo, A. Montanino, C. Olivieri, A. Fortunato, Automatic generation of statically admissible stress fields in masonry vaults, Journal of Mechanics of Materials and Structures 14 (5) (2019) 719–737.