Closest-Fit, Compression-Only Solutions for Freeform Shells

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Summary

This paper introduces a powerful computational design and analysis method for shells, based on funicular networks. Based on Thrust Network Analysis, an optimization approach is presented, which finds the closest fit, compression-only solution to an arbitrary input surface for given network topologies,. This research offers new possibilities for the design of efficient, but exciting shells which start to visually blur the forms which have been associated to compression-only and freeform shell. Furthermore, this method provides the foundation for a fully three-dimensional equilibrium analysis method for historic masonry vaults with complex geometries. The paper gives new insight in the force dependencies in 3D funicular networks, which serve as a basis for the nonlinear solving procedure presented. Through several examples, the power of this novel research is demonstrated.

Keywords: *Compression-only shells, freeform surface structures, structural optimization, Thrust Network Analysis, Force Density Method.*

1. Introduction

Compression-only shells have efficient structural forms which minimize material use and render the use of reinforcement obsolete or severely reduce it. Historically, physical design techniques such as hanging cable nets (Antoni Gaudi, Frei Otto) or cloths (Heinz Isler) have been used as powerful, but time and cost intensive form finding approaches for compression shells. Different computational approaches have been presented, in order to ease the design process of such three-dimensional funicular networks [1,2].

In general, three-dimensional funicular systems are indeterminate to a high degree, i.e. a given set of applied loads generates several solutions. A unique solution can be enforced by assigning force densities to the network [3]. Recently, Thrust Network Analysis (TNA) has been introduced as a method, to control the force densities directly in order to fully exploit all degrees of indeterminacy in design and analysis [4,5]. Using the concept of reciprocal diagrams [6], the force densities, hence the shape can be controlled explicitly, and thus be used for direct design explorations. This forward design approach provides new insights in the structural behaviour of funicular systems, and enables a formal richness in design, far beyond typical inverted “hanging” forms [7]. But it is not easy to steer the shape of the funicular network in the direction of a desired form by the means of the reciprocal diagram without experience.

This paper describes a numerical method that allows the efficient application of the concepts of TNA to design and analysis of compression-only shells and vaults. The direct method of TNA is embedded into a parameter search, in order to find the closest-fit compression solution of a network with a given planar projection, for a given surface and given vertical loading. The main applications for this method are the design rationalization of a given shape or the assessment of the safety of historic masonry vaults.

The structure of this paper is as follows. Section 2 briefly summarizes the theoretical background of the approach, especially the relation between the reciprocal diagram and force densities. Section 3 describes the method and its computational setup in detail. Section 4 finally shows examples that demonstrate the power of the presented approach, not only for the design of free form shells, but also for the equilibrium analysis of historical unreinforced masonry vaulted structures with complex geometries.

2. Controlling force densities

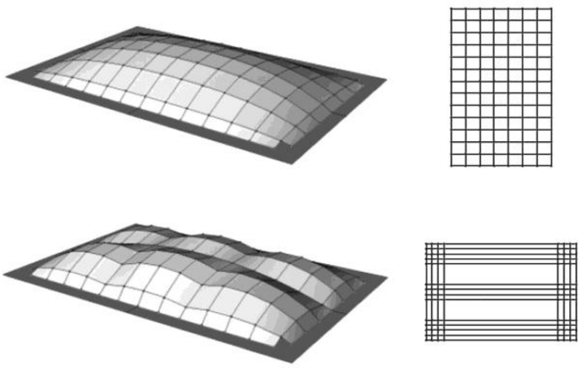

Fig. 1: Redistribution of the internal forces, controlled by reciprocal force diagrams [5].

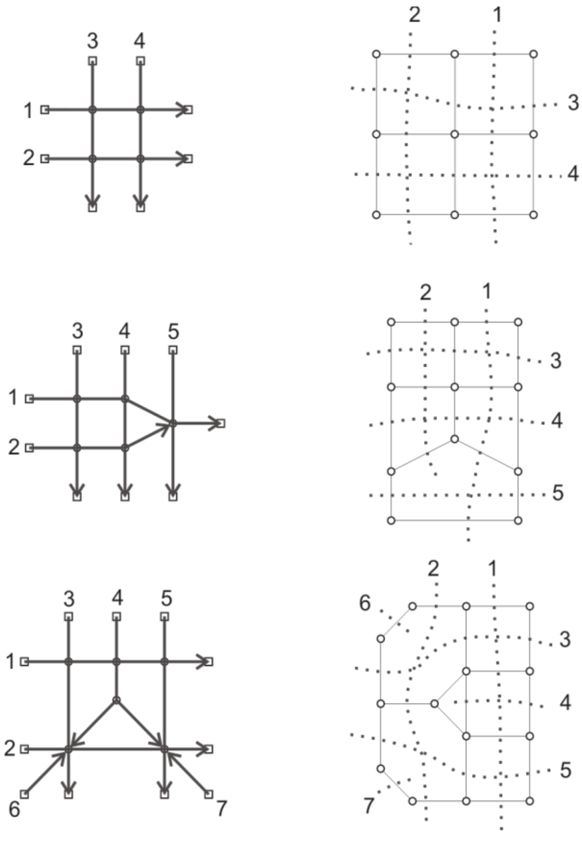

Fig. 2: The dependency of the force densities in the form diagram (left) can be understood through visual inspection of the reciprocal diagram (right).

Three-dimensional funicular networks generally have a high degree of indeterminacy; forces can be redirected and attracted in the different directions in space allowed in the topology. This was clearly shown by [4] by means of horizontal reciprocal force diagrams for the case of vertical loading only. These reciprocal force diagrams represent the in-plane equilibrium of the form diagram, i.e. the horizontal projection of the 3D funicular network. The existence of different force diagrams for the same form diagram correspond to different internal force distributions, and hence different compression-only solution for the same set of vertical loads (Fig. 1).

Reciprocal diagrams furthermore visualize geometrically how local changes influence global internal force equilibrium (Fig. 2). This intuitive reading of the indeterminacy comes from the understanding of the reciprocal constraints of parallelity between form and force diagram [5,6].

Translated to the Force Density framework [3], this means that allowed reciprocal diagrams represent combinations of force densities q which maintain equilibrium in compression-only, while keeping the horizontal projection of the network fixed. The direct relation between the force densities q and the geometry of the reciprocal diagrams, l_H^*, is the following

$$q = L^{-1}s = L_H^{-1}s_H = L_H^{-1}l_H^* \qquad (1)$$

with l and s the branch lengths and axial forces of the equilibrium network, l_H and s_H their respective horizontal components, and l_H^* the reciprocal branch lengths.

From Eq. 1, it is clear that not all force densities q can be chosen freely, as not all reciprocal branch lengths are independent (because of the reciprocal constraints). This can be seen from inspection of Fig. 2; only a specific set of reciprocal branch lengths - and thus equivalently force densities- can be chosen independently without violating the reciprocal constraints –or equivalently (horizontal) equilibrium.

The intuitive understanding of force density dependency, as visualized in Fig. 2, can be formalized for networks of any topology. It is again the reciprocal diagram which provides the insight, but using another geometric analogy. The geometrically allowed variations of the reciprocal, i.e. those that keep its branches parallel to the respective branches of the form diagram, are related to the inextensible mechanisms of the reciprocal diagram, considered as a 2D pin-jointed, bar-node structure which is properly restrained, i.e. externally determinate. These mechanisms can be identified using a matrix analysis procedure which uses Gauss Jordan Elimination (GJE) [8]. Here, this procedure thus directly defines a possible set of branches whose force densities q_{indep}, or equivalently (horizontal) thrusts s_H (= l_H^*), can be chosen independently.

3. Optimization Method

The optimization method presented in this paper uses the matrix formulation of the Force Density Method [3] enhanced with new extensions of Thrust Network Analysis (TNA) [5].

Based on the new approach for identifying the force dependencies between branches in equilibrium networks (see above), an efficient optimization routine has been developed to find the closest compression-only network solution to an arbitrary surface.

3.1 Overview

The set of independent force densities represent the only necessary, and also the minimal amount, parameters to describe and control the 3D equilibrium of a network. The closest-fit problem can thus be reduced to finding the set of q_{indep} which minimize the global objective

$$f(z) = \left\| z - z^T \right\| \quad (2)$$

with z^T the z-coordinates of the nodes of the form diagram projected on the target surface.

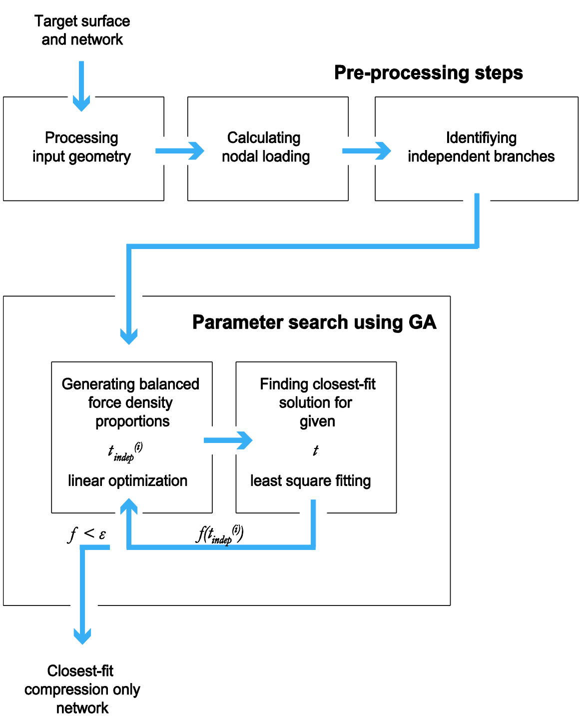

Fig. 3: Flow diagram of the computational approach.

Fig. 3 shows the flow diagram of the optimization approach which includes some geometrical and topological pre-processing steps (Section 3.2) and an iterative parameter search (Section 3.3).

3.2 Pre-processing steps

As the topology stays unchanged in the nonlinear optimization, the computations related to the topology of the network can be performed separately first.

3.2.1 Processing input geometry

The topology of the network is captured with a branch-node matrix Cs. From the geometry of the network (x,y) and the topology of the network (Cs), the reciprocal constraints matrix K is then constructed (see 3.3.1).

3.2.2 Calculating nodal loading

Since the optimization searches for the closest-fit solution to a given target surface, the height field z^T, obtained from the target surface for the fixed projection (x,y), can be used to have a good approximation of the 3D tributary area for generating the lumped nodal loads.

3.2.3 Identifying independent branches

With the matrix analysis mentioned above, a possible set of independent branches is identified, whose force densities are the k independent parameters of the problem.

3.3 Parameter search

The parameter search is conducted using a genetic algorithm (GA) because the problem is discrete non-convex, and very spiky. At each iteration, the fitness is calculated implicitly using two fast optimization steps, described in the next two subsections.

3.3.1 Generating balanced force density proportions

Because the applied loading is vertical, they do not appear in the reciprocal diagram [4]. This means that it can be scaled without affecting the components of the applied loading, and thus that the force densities can be written as

$$q = \frac{1}{r} t \quad (3)$$

where t have the proportional relation of the force densities, but need to be scaled with the scalar $(1/r)$ to render the real values of the force densities q.

Instead of directly using the q_{indep} as the parameters of the problem, the t_{indep}, together with r, are used in order to limit the search space of the GA to $t_{indep} = [0,1]$.

The properties of the reciprocal diagram can be used to find all the t corresponding to the inputted $t_{indep}^{(i)}$. The equations representing the reciprocal constraints between the branch vectors of the force network u and v and its reciprocal diagram u^* and v^* can be written in matrix form [5]

$$Kr^* = 0 \quad (4)$$

with $r^* = [\, u^* \mid v^* \mid t \,]^t$ and

$$K = \left[\begin{array}{cc|c} C^t & 0 & \multirow{2}{*}{0} \\ 0 & C^t & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{I_{2m}}} & -U \\ & & -V \end{array}\right] \quad (5)$$

in which I_{2m} is the identity matrix of size $2m$, C the first n_i columns of Cs, with n_i the number of inner nodes of the form diagram, and U and V the diagonal matrices of u and v respectively. The first $2n_i$ rows of K are the dual constraints equations and the last $2m$ rows of K the parallelity constraints equations.

The dependent t_{dep}, related to the inputted $t_{indep}^{(i)}$, could be found directly using linear algebra, but a non-negative set of $t_{indep}^{(i)}$ does not guarantee non-negative t_{dep}, which is a necessary condition for a convex reciprocal, or alternatively a compression-only solution [5]. In order to avoid this, the following linear optimization is introduced

$$\min_t \; t \quad s.t. \quad \begin{cases} Kr^* = 0 \\ t_{dep} \geq 0 \\ t_{indep} \geq t_{indep}^{(i)} \end{cases} \quad (6)$$

This optimization pushes the t_{indep} to be as close to $t_{indep}^{(i)}$ without the t_{dep} becoming negative.

3.3.2 Finding closest-fit solution for given t

Using the set of t obtained in 3.3.1, the closest-fit compression-only network to the target surface is found with the following least-square data fitting problem with linear constraints

$$f = \min_{z,r} \left\| z - z^T \right\| \quad s.t. \quad \begin{cases} Dz - rp = 0 \\ r > 0 \end{cases} \qquad (7)$$

with p the loading vector, and $D = C^{\,t}TCs$ where T is the diagonal matrix of t. The optimization renders the closest-fit solution z and the overall scale of the reciprocal, $1/r$. The force densitites q are then directly found using Eq. (3). The cost f of the optimization (7) is furthermore directly the fitness evaluated in the GA at that iteration for the $t_{indep}^{(i)}$.

3.4 Implementation

A fully working prototype implementation has been developed in RhinoScript [9] for the import and export of the geometry, and in Matlab for the computation, using the following routines from the optimization toolbox [10]:

- *ga* for the global parameter search, based on genetic algorithms (3.3);
- *linprog* for the linear optimization to find a non-negative set of t for a choice of t_{indep} (3.3.1); and
- *lsqlin* for the least-squares data fitting of the z to the target surface z^T.

4. Results and discussion

This paper will show examples that demonstrate the power of the presented approach, not only for the design of freeform shells, but also for the equilibrium analysis of historical unreinforced masonry vaulted structures with complex geometries.

4.1 Design of freeform, funicular shell

The examples in Figs. 4 and 5 serve as a simple proof of concept of the optimization method, convincingly demonstrating its power. convincingly demonstrating its power.

For a freely formed surface and randomly generated form diagram, the figures show

a) the closest-fit, compression-only solution;

b) a proportional distance map, showing the very good matching between the compression-only network and the target;

c) a pipe diagram proportional to the magnitude of the branch forces, visualizing how forces are flowing and being attracted along certain branches; and

d) the reciprocal force diagram showing the distribution of force densities, or equivalently horizontal thrusts.

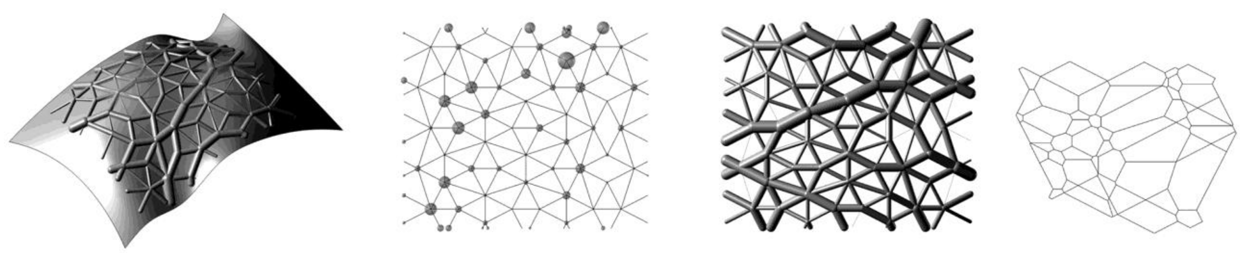

Fig. 4: For a freely formed target surface and random network: a) closest-fit, compression-only solution; b) distance map; c) pipe diagram proportional to the branch forces; and d) the reciprocal force diagram showing the distribution of force densities, or equivalently horizontal thrusts.

The second pattern shown in Fig. 5 starts from the first pattern, but adds "open edge arches" along the boundaries. These constrain the flexibility of the pattern, as witnessed in a reduction of degrees of freedom (see Table 1). Still, there is very good matching.

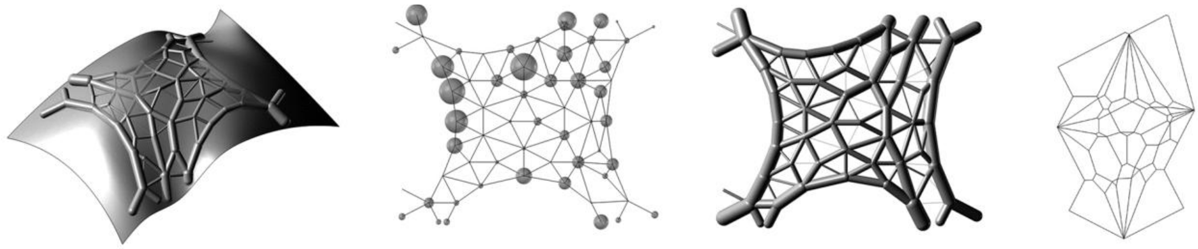

Fig. 5: Starting from the form diagram of the example in Fig. 4, "open edge arches" are added to the pattern, reducing the degrees of freedom of the network topology.

From Table 1, it can be seen that the implemented optimization method is very efficient. The two examples in Figs. 4 and 5 found the best-fit solution in less than a minute. The results were obtained in Matlab using a Dell Precision T7400, Intel Xeon CPU X5460 3.16GHz, 8.00 GB RAM. The table also shows for these different patterns their degree of freedom k, i.e. the number of branches whose force densities can be controlled independently.

Table 1: For the examples in Figs. 4-6: total number of branches, degrees of freedom of the network patterns, and the solving times of the optimization.

Fig.	Number branches m	Degrees of freedom k	Solving time (sec)
4	171	63	37
5	127	35	52
6	528	51	568

The pattern of the examples in Figs.4 and 5 is quite flexible because it has branches in several directions, and most importantly in the direction of "creases". In order to approximate a surface well, the used pattern indeed needs to have branches aligned in the directions in which creases should be generated, i.e. more force should be attracted.

From this example, it is clear that freeform shells can be designed easily by finding the closest-fit solution automatically. The resulting force distribution suggests directly which aspects of the geometry of the target surface can be improved to have a more efficient shell, i.e. by reducing the extreme forces.

A direct form finding approach can be very powerful. It furthermore forces the designer to consciously consider the forces in the system, during the interactive form finding (ref iass last). As important shortcoming though, such an approach cannot check if a compression-only solution can be found within the same structure under asymmetric loads. The method presented in this paper now allows to also easily consider –because of the speed of calculating- a series of asymmetric loading cases already in the initial design stage.

A very revealing and interesting conclusion of this research is that it suggests that a compression-only solution can be fitted to *any* freely formed target surface as long as it does not curl back onto itself and has no "puddles". The supports also have to be on the "outside" of the network. The question thus is no longer *if* it is possible to realize a freeform shell with such properties, but rather *at what cost*, i.e. how do the forces need to be redistributed internally in order to achieve a certain three-dimensional shape. Of course, this is only possible when either a very dense and highly indeterminate, hence flexible, force network topology is used, or alternatively, the choice or generation of network topology has been informed by the target surface's topography and curvature. As an example, a good fit will only be found for a shape with a crease, if force lines run along it.

4.2 Equilibrium analysis of masonry vaults

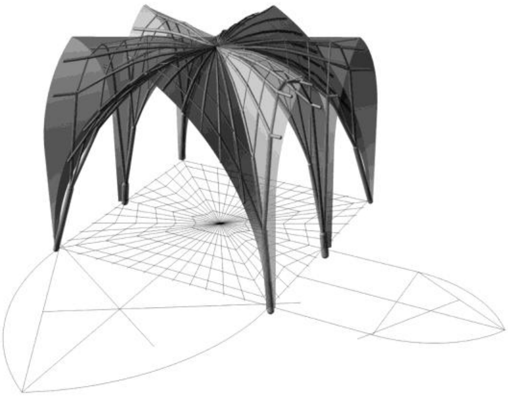

Fig. 6: A thrust network following the middle surface of a sexpartite Gothic vault.

The field for which these new developments will have the most impact is three-dimensional equilibrium analysis of historic vaults in unreinforced masonry [11,12].

These vaults have often complex geometries and a forward approach is unrealistic to be able to get to a force distribution resulting in a thrust network fitting within the geometry of the vaults. Furthermore, in historic masonry structure a discrete approach where force paths can be chosen is highly relevant due to their often cracked and displaced nature. Fig. 6 shows a fully 3D thrust network mapped almost perfectly to the middle surface of the complex sexpartite geometry.

5. Conclusion and future work

This paper introduced a powerful optimization approach, based on extensions of Thrust Network Analysis, which finds the closest-fit, compression-only solution to an arbitrary input surface for given network topologies. The large potential for both the design of freeform (and efficient) shells and for fully three-dimensional equilibrium analysis of historic masonry vaults have been demonstrated through a series of examples. An important conclusion of the research presented in this paper is that by controlling the high degree of indeterminacy of 3D funicular networks, compression shells can be generated which have not been imagined before.

Next steps in this research include the coupling of this solving procedure to the choice or generation of force patterns in order to improve the fitting, and implementing additional constraints necessary to further develop this to a full-fledged lower-bound analysis approach for historic masonry vaults, a field in desperate need for such tools.

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