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Form-finding explorations through geometric transformations and modifications of force polyhedrons

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Abstract

This paper introduces a new structural modeling methodology to generate equilibrated, discrete three-dimensional structures—with both compression and tension elements—through geometric transformations and modifications of *force polyhedrons*. By exploring the geometric properties of *polyhedral reciprocal diagrams* used in three-dimensional *graphic statics*, the proposed approach provides a significant amount of control of spatial structures during conceptual design. The presented design examples will demonstrate how this method can be used to discover new three-dimensional structural typologies without any biases towards conventional solutions. They also provide an alternative design strategy for two-dimensional problems.

Keywords: three-dimensional graphic statics, polyhedral reciprocal diagrams, discrete spatial structures, generative design, form finding, spatial equilibrium modeling

1. Introduction

With conventional tools, it is cumbersome to model discrete structures in three dimensions. A designer has to model the structure manually through construction planes using a mouse that is already constrained to the plane of the monitor screen, or generate one through a parametric procedure. Equilibrium of the resulting structure needs to be checked and sized using finite element analysis software. Even if an equilibrium of a spatial structure is achieved, it is difficult to make modifications to the geometry in an interactive manner; a substantial remodeling and reanalysis are required for any design changes. Subsequently, it is difficult to explore new spatial typologies efficiently, and the resulting spatial structures are often arrays or accumulation of two-dimensional solutions.

In order to explore a wide range of spatial structures with sufficient amount of control, this paper proposes a new modeling methodology that utilizes the polyhedral reciprocal diagrams as the generator of structural form. By using geometries of force polyhedrons, the entire structure can be easily controlled and modified without breaking its spatial equilibrium. Generating the structural form by manipulating the force diagram also means that the structural typology is an unknown at the start, resulting in feasible designs that are not biased towards known solutions or predefined typologies. The proposed method expands on previous research in this topic (Akbarzadeh *et al.* [1], [2], [4]) by incorporating both compression and tension elements into a variety of polyhedral transformations. In addition to generative design of three-dimensional structures, the polyhedral methods presented in this paper can also be used to address two-dimensional problems.

2. Background

In two-dimensional graphic statics, manipulation of the geometry of *force polygons* has been used in a range of design applications: force-constrained form finding (Allen and Zalewski [5], Van Mele *et al.* [13], Fivet and Zastavni [10]), optimal search through geometric optimization of force diagrams (Beghini A. *et al.* [7], Beghini L. *et al.*[8]), or interactive design of free-form shell structures using projected horizontal force diagrams (Block [9], Rippmann *et al.* [12]). These applications are based on modifications of force diagrams, which need to be preceded by a predetermined form of a specific structural typology.

Akbarzadeh *et al.* have shown how recursive subdivision of a global force polygon [1] or polyhedron [4] can generate diverse funicular forms with different topological properties for the same given boundary condition. While successful in demonstrating how manipulation of force diagrams can be used as a means of generative structural design, the subdivision method is based on only one polyhedral operation and, as presented, limited to compression-only or tension-only structures.

3. Methodology

This section summarizes the concepts of polyhedral reciprocal diagrams that are used in this paper.

Conventionally, an element of a dual diagram and a reciprocal diagram are suffixed with an asterisk (*) and the perpendicular symbol (⊥), respectively. However, as the variables used in this paper are explicitly introduced and defined as elements of the reciprocal force polyhedron, the perpendicular symbol will be omitted for simplicity.

3.1. Reciprocal polyhedral diagrams

In polyhedral reciprocal diagrams used in three-dimensional graphic statics, the equilibrium of the external forces or a single node of an equilibrated structure is represented by a closed force polyhedron or *polyhedral cell* (Rankine [10], Akbarzadeh *et al.* [2]). For a node $\mathbf{v}_\mathrm{i}$ with *e* number of external forces and/or connected members, a face f_j of the reciprocal force polyhedron corresponds to the *j*th external force or the axial force in the *j*th member, $\mathbf{f}_\mathrm{j}$, which is oriented perpendicular to this face (Figure 1a-b). The areas A_j and normals $\hat{\mathbf{n}}_\mathrm{j}$ of these faces are the magnitudes and directions of the corresponding forces in the form diagram, $\mathbf{f}_\mathrm{j} = A_\mathrm{j} \cdot \hat{\mathbf{n}}_\mathrm{j}$.

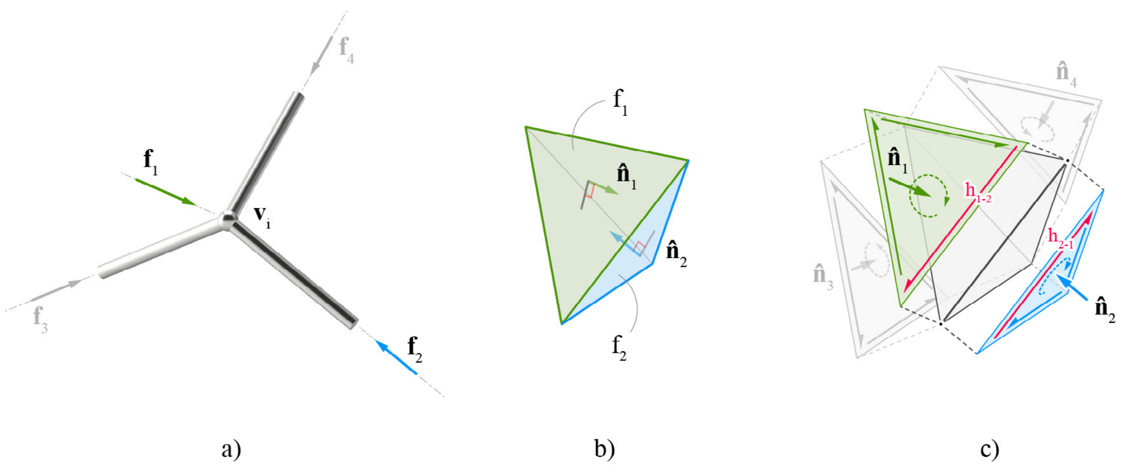

Figure 1 : Overview of a) node $\mathbf{v}_\mathrm{i}$ of a structure in equilibrium; b) force polyhedron of that node; and c) that force polyhedron exploded with directions of face normals and half-edges shown.

In a reciprocal force polyhedron where each face is represented by an oriented mesh (Figure 1b), every pair of adjacent half-edges is equal in length and opposite in directions (for example in Figure 1c, the two half-edges h_{1-2} and h_{2-1}). Therefore, if the normal and half-edge directions of one face of a force polyhedron are known or assigned, the half-edge directions of the remaining faces can be determined (Akbarzadeh *et al.* [2]). In general, the face normals of a single convex force polyhedron are unified (collectively referred to as the *polyhedral direction*); they all point either away from the center of the polyhedron, or towards the center of the polyhedron (Akbarzadeh [5]).

3.2. Compression and tension

A force $\mathbf{f}_\mathrm{j}$, acting on node $\mathbf{v}_\mathrm{i}$, can be either a compression force or a tension force; it is a compression force if it is pushing onto the node, or a tension force if it is pulling away from the node. In three-dimensional graphic statics, the directions of the forces $\mathbf{f}_\mathrm{j}$ are the same as the normals $\hat{\mathbf{n}}_\mathrm{j}$ of their corresponding face in the force polyhedron (Akbarzadeh [5]). Therefore, a single force polyhedron can represent the equilibrium of various combinations of compression and tension forces.

The interpretation of the force in the *j*th structural member at node $\mathbf{v}_\mathrm{i}$ as either compression or tension can be made by comparing the polyhedral face normal $\hat{\mathbf{n}}_\mathrm{j}$ and the orientation of that member. Locally defined for each node $\mathbf{v}_\mathrm{i}$, the orientation of the *j*th member can be represented by vector $\mathbf{e}_{\mathrm{i,j}}$ with the tail of the vector at the node. If vectors $\hat{\mathbf{n}}_\mathrm{j}$ and $\mathbf{e}_{\mathrm{i,j}}$ are in the same direction, the force in the corresponding member is positive, so in tension; if the vectors are in opposite directions, the force in the corresponding member is negative, so in compression. This method of interpreting the force in a member as either compression or tension through visual inspection of the form and force diagrams based on a consistent directional convention, is similar to the one used in typical two-dimensional graphic statics (Allen and Zalewski [6], Van Mele and Block [14]). Figure 2 illustrates three possible configurations of compression and tension members for a single force polyhedron, and the respective orientations of the members.

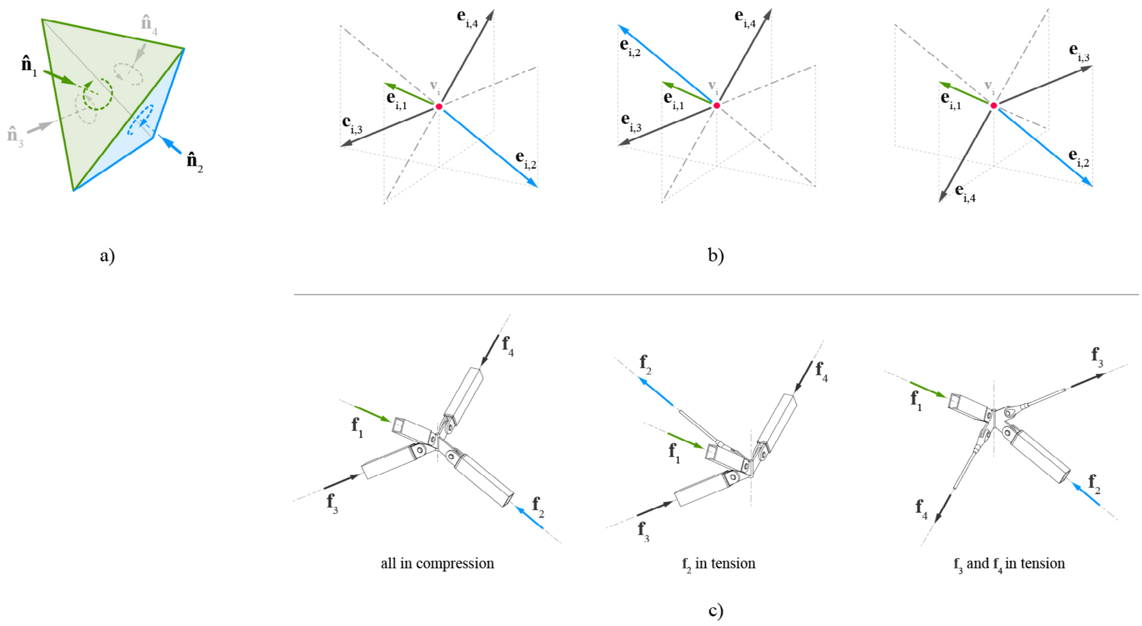

Figure 2 : Multiple configurations of compression and tension elements are possible for the same force polyhedron: a) force polyhedron for node $\mathbf{v}_\mathrm{i}$; b) orientation of the members as vectors $\mathbf{e}_{\mathrm{i,j}}$; and c) the direction and type of forces in the members.

3.3. Global and nodal force polyhedrons

A polyhedral force diagram for a spatial structure in equilibrium can be decomposed into two sets of force polyhedrons: a *global force polyhedron* (GFP) that describes the equilibrium of externally applied and reaction forces, and an aggregation of *nodal force polyhedrons* (NFP) that describe the local equilibriums of forces coming together at the nodes (Akbarzadeh *et al.* [3]). The GFP for an equilibrated structure is closed, and therefore its polyhedral direction is known at the start. Consequently, the polyhedral directions of all NFPs can be determined since the normal directions of the faces belonging to the GFP are known (Figure 3c).

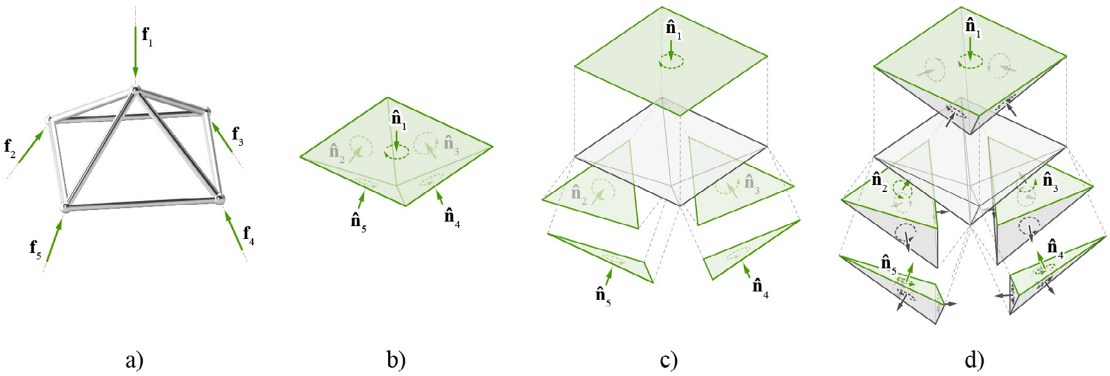

Figure 3 : a) A polyhedral frame with external forces at its vertices; b) the corresponding GFP with its face directions; c) exploded GFP; and d) the five NFPs with corresponding polyhedral directions.

3.4. Workflow

First step of the proposed methodology is to formulate the boundary conditions by defining the magnitudes and locations of the applied forces, and the location of the supports. The GFP can then be geometrically constructed to obtain the reaction forces (Akbarzadeh [3]). For an indeterminate system of external forces, the user will need to determine an initial distribution of reaction forces to start the transformative process, which can be modified later. From a set of polyhedral transformations and modifications, which will be described in Section 3.5, either a polyhedral vertex, face or cell—individually or in groups—can be chosen by the user to apply the geometric operation. These operations can be applied iteratively in any order, until a desired design criteria is reached.

After each manipulation, a feasible geometry of the polyhedral frame can be computed using an algorithm that imposes duality and perpendicularity (Akbarzadeh *et al.* [2]), i.e. the reciprocity between the polyhedral form and the force diagrams.

The described workflow is summarized in the flowchart in Figure 4.

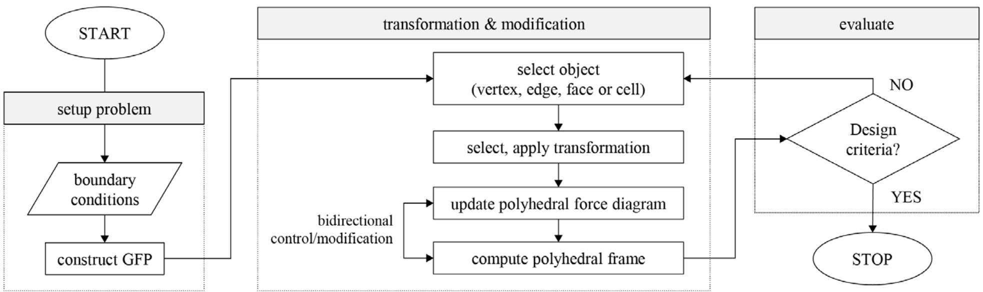

Figure 4 : Flowchart of the proposed computational method.

3.5. Polyhedral operations

Figure 5 summarizes the polyhedral edits used in this paper. This summary does not contain all possible transformations and modifications, and will be expanded in future research. In general, all edits are geometrically constrained to output polyhedral solutions, so with planar faces.

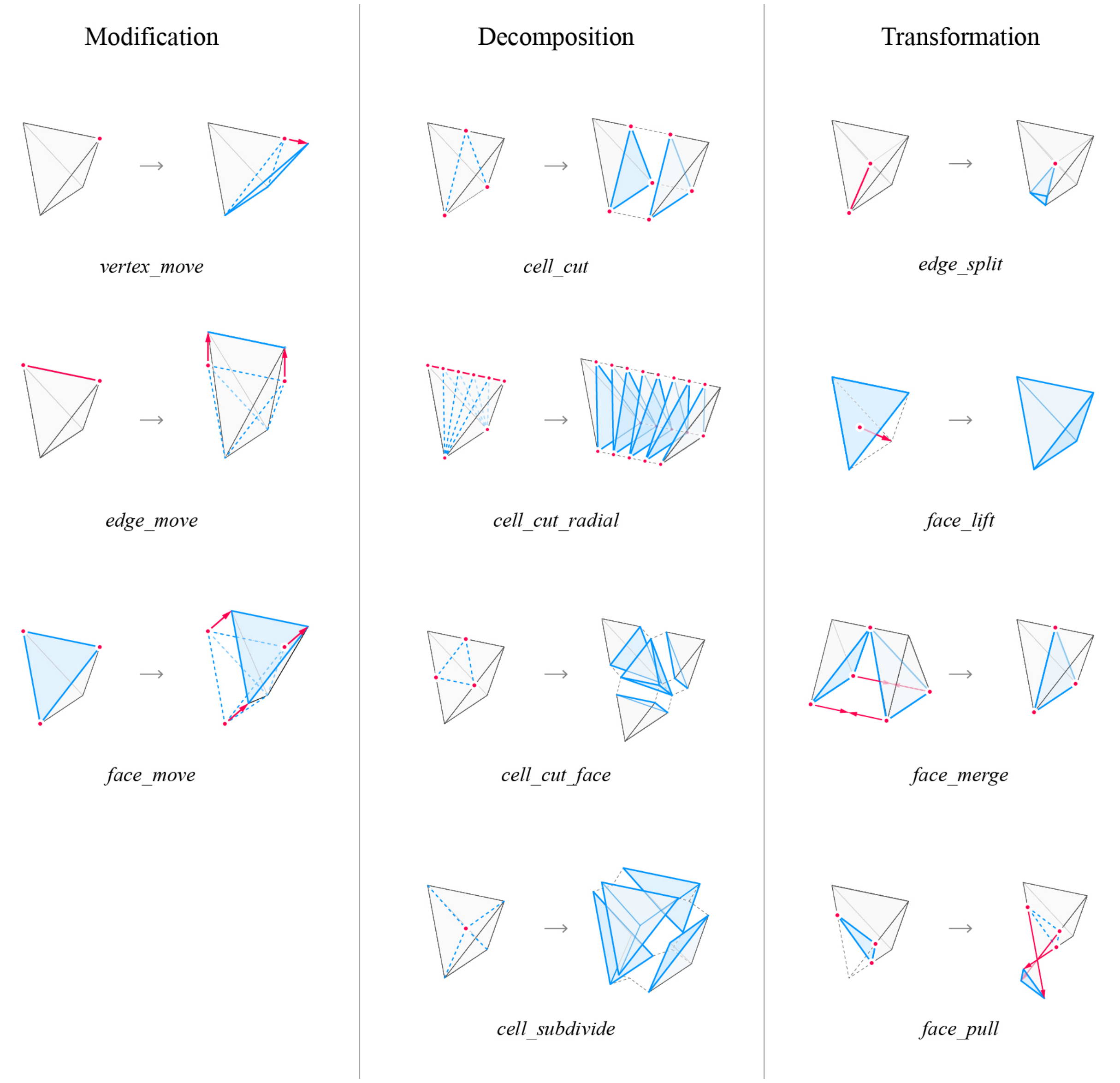

Figure 5 : Three categories of polyhedral edits: a) modification; b) decomposition; and b) transformation.

4. Results

This section presents variations for three design scenarios generated using the proposed method: a horizontally spanning structure with one point load and three vertical (two roller and one pinned) supports (Figure 6); a vertical cantilever structure with three pinned supports and a horizontal point load (Figure 7); and a bay of a large roof with a single tower support at the center and a suspended roof (Figure 8).

4.1. Example 1: Horizontally spanning structure

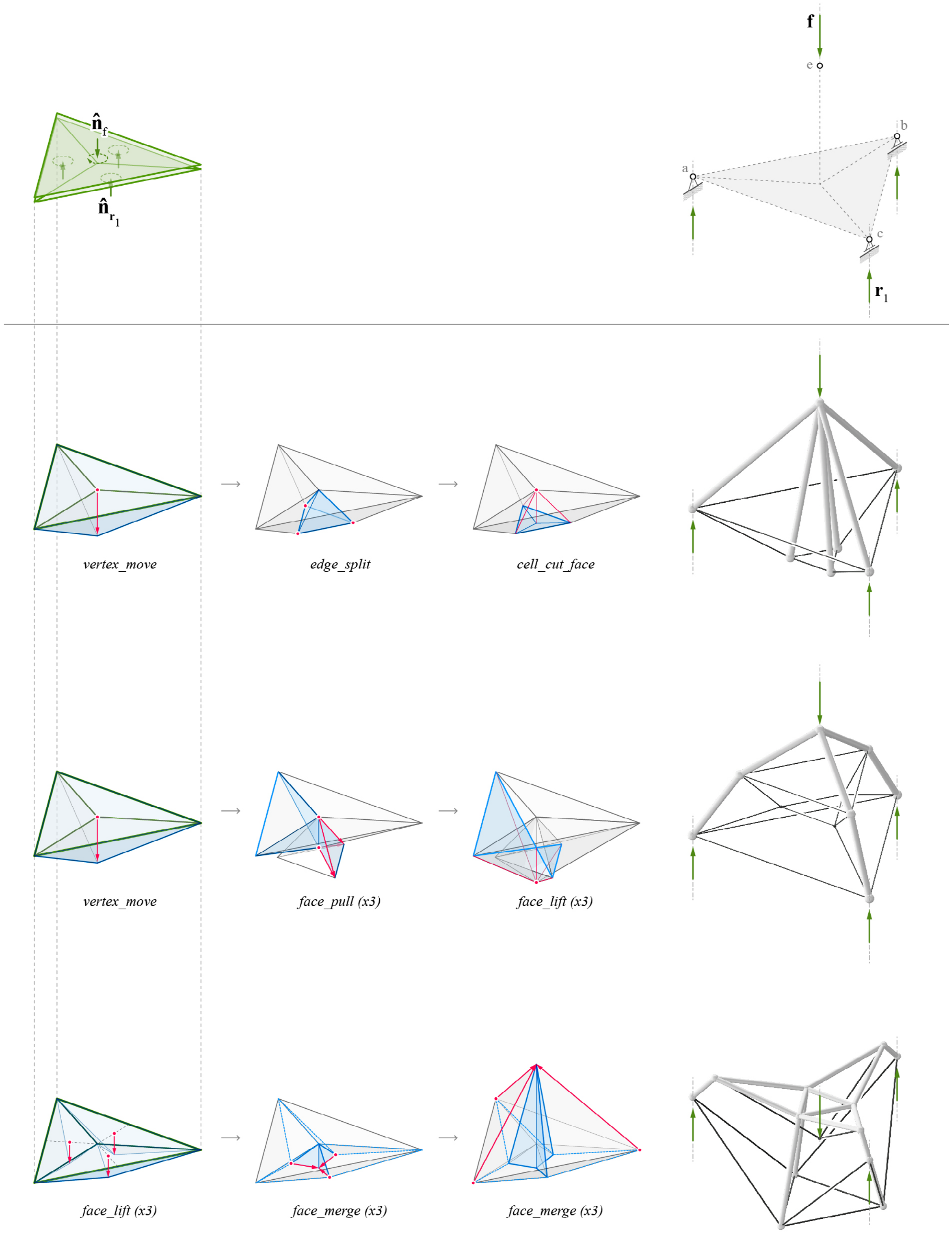

Figure 6 : Design example 1: a horizontally spanning structure with three supports.

4.2. Example 2: Vertical cantilever

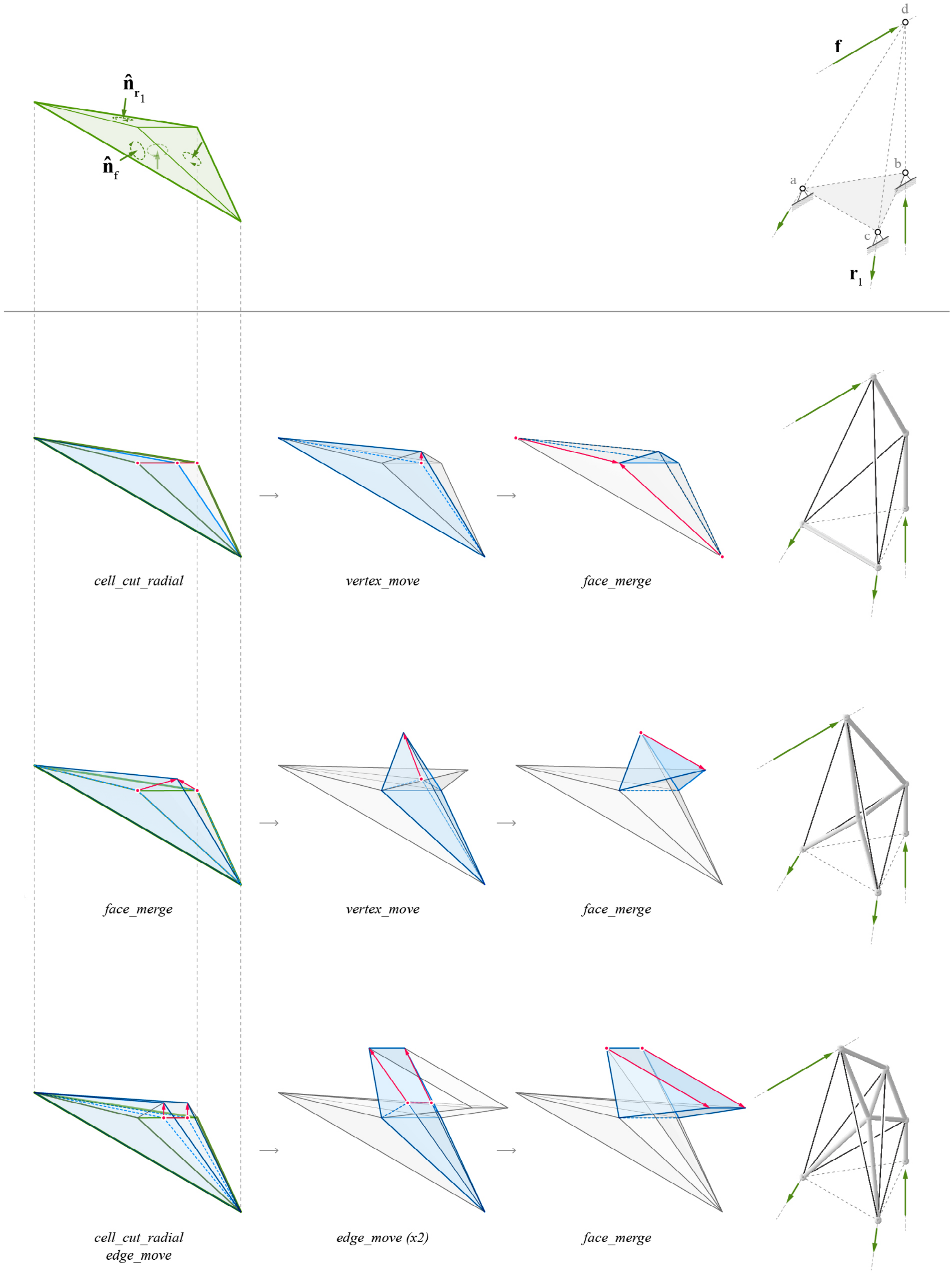

Figure 7 : Design example 2: a vertical cantilever with horizontal point load.

4.3. Example 3: Tower and suspended roof

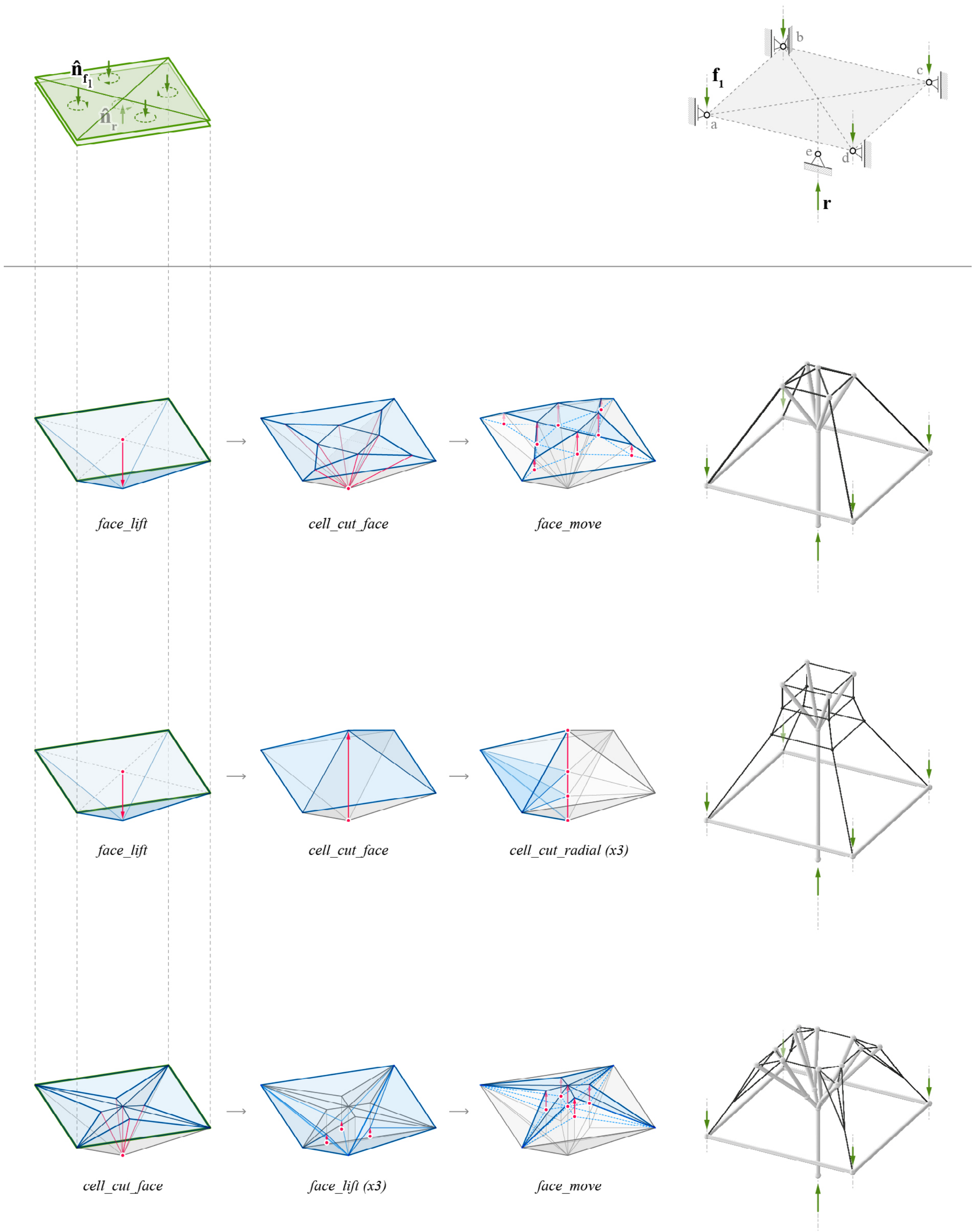

Figure 8 : Design example 3: a bay of a suspended roof structure with a central tower.

4.4. Application to 2D problems

Two-dimensional equilibrium is a special case of three dimensional equilibrium where one of the axes is perpendicular to the plane of the two-dimensional structure. Because the methods presented in this paper are based on manipulating force polyhedrons to design three-dimensional structures, the same strategies can also be used to design two-dimensional structures. Consider the force polyhedron in Figure 9a. Polyhedral face f_1 is oriented on plane P_1, which is parallel with the viewing plane V of the structure shown in Figure 9c. By orienting another face of the force polyhedron f_2 onto plane P_2 that is parallel with plane P_1, and constraining the remaining faces of the force polyhedron to be perpendicular to planes P_1 and P_2, two faces f_1 and f_2 will consequently have equal areas and opposite normal directions (Figure 9a). A_1 and A_2 multiplied by the respective face normals represent equal and opposite forces $\mathbf{f}_1$ and $\mathbf{f}_2$ acting on node $\mathbf{v}_i$ in the structure in Figure 9c, and the sum of the forces acting out of plane V is zero. The lengths of the half-edges h_{j-V} in Figure 9b are proportional to the magnitudes of the horizontal forces $\mathbf{f}_j$ in Figure 9c, subject to a scale factor ζ, which is the vertical extrusion length of the polyhedron. These half-edges are vectors equivalent to the force polygons used in typical two-dimensional graphic statics.

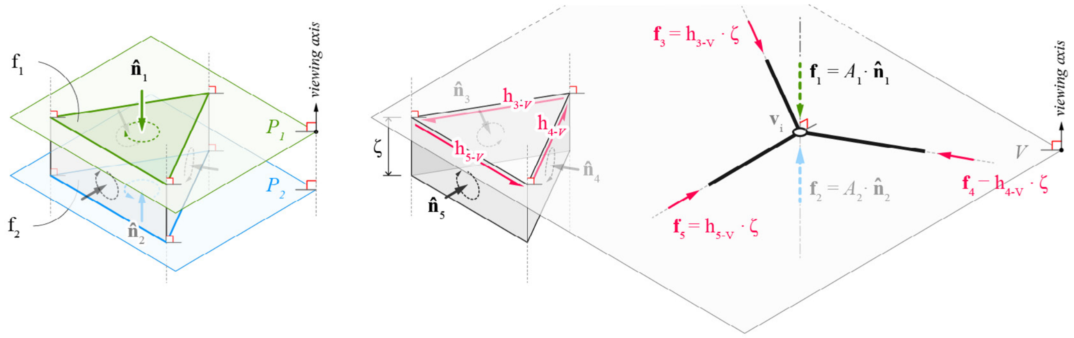

Figure 9 : a) A force polyhedron with two parallel faces and all other faces in perpendicular orientation; b) the edges of vertical faces form a closed polygon on plane V; and c) the corresponding two-dimensional structure.

The equivalent polyhedrons for two-dimensional structures can be edited with the same rules as in the design examples shown in Sections 4.1 through 4.3. Design examples of two-dimensional structures will not be presented in this paper. However, in order to demonstrate its principle, Figure 10 shows how the same manipulations can be used to generate a two-dimensional, discrete Michell truss.

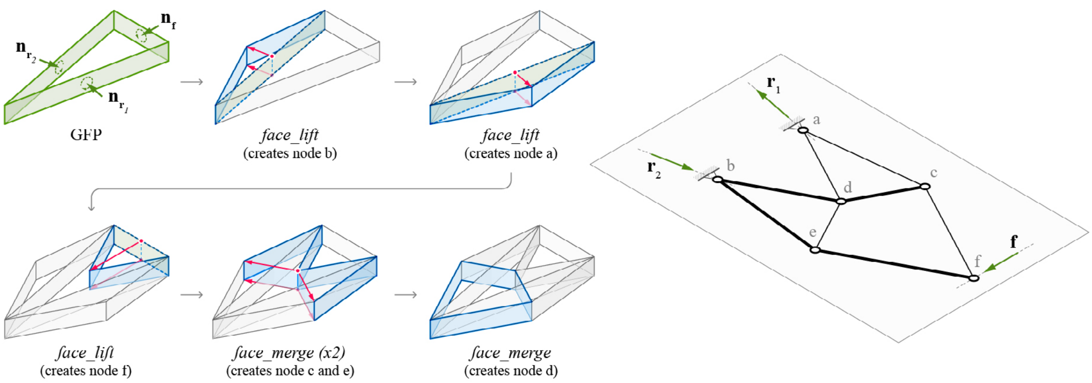

Figure 10 : Sequence of manipulations to generate a discrete Michell truss.

5. Conclusion

This paper presented a new approach to modeling discrete three-dimensional structures with both compression and tension elements through geometric operations—transformations and modifications—of force polyhedrons. The methods increase the amount of control the user has over discrete three-dimensional structures during and after the transformative design processes. It also improves the potential of discovering new spatial topologies and typologies since the exact structural form of the eventual solution is always unknown. Because two-dimensional structures are special cases of three-dimensional structures, the polyhedral strategies demonstrated in this paper can also be used to address two-dimensional problems, as shown in Section 4.4.

Topics for future research may include: 1) developing and visualizing more sophisticated geometric constraints on the force polyhedrons to actively guide the transformations; 2) automating the application of transformation; and 3) designing algorithms that guide the automation process towards a design criteria, such as minimization of material, length and number of elements, etc.

References

[1] Akbarzadeh M., Van Mele T. and Block P., Compression-only form finding through finite subdivision of the external force polygon. *Proceedings of the IASS-SLTE Symposium 2014*, Brasilia, Brazil, 2014.

[2] Akbarzadeh M., Van Mele T. and Block P., On the equilibrium of funicular polyhedral frames and convex polyhedral force diagrams. *Computer-Aided Design*, 2015; **63**; 118-128.

[3] Akbarzadeh M., Van Mele T. and Block P., 3D Graphic Statics: Geometric construction of global equilibrium. *Proceedings of the IASS Symposium 2015*, Amsterdam, Netherlands, 2015.

[4] Akbarzadeh M., Van Mele T. and Block P., Three-dimensional compression form finding through subdivision. *Proceedings of the IASS Symposium 2015*, Amsterdam, Netherlands, 2015.

[5] Akbarzadeh M., *Three-dimensional Graphical Statics using Polyhedral Reciprocal Diagrams.* PhD thesis, ETH, Zürich, Switzerland, 2016 (forthcoming).

[6] Allen E. and Zalewski W., *Form and Forces: Designing Efficient, Expressive Structures.* John Wiley Sons, New York, 2010.

[7] Beghini A. and Baker W. F., On the layout of a least weight multiple span structure with uniform load. *Structural and Multidisciplinary Optimization*, 2015; **52**: 447–457.

[8] Beghini L. L., Carrion J., Beghini A., Mazurek A. and Baker W. F., Structural optimization using graphic statics. *Structural and Multidisciplinary Optimization*, 2013; **49**; 351-366.

[9] Block P., *Thrust Network Analysis: Exploring Three-dimensional Equilibrium*. PhD thesis, MIT, Cambridge, MA, USA, 2009.

[10] Fivet C. and Zastavni D., Constraint-based graphic statics: new paradigms of computer-aided structural equilibrium design. *Journal of the International Association for Shell and Spatial Structures, 2013;* **54** (178); 271-280.

[11] Rankine M., Principle of the equilibrium of polyhedral frames. *Philosophical Magazine*, 1864; **27** (180); 92.

[12] Rippmann M., Lachauer L. and Block P., Interactive vault design. *International Journal of Space Structures*, 2012; **27** (4**)**; 219-230.

[13] Van Mele T., Lachauer L., Rippmann M. and Block P., Geometry-based understanding of structures. *Journal of the International Association of Shell and Spatial Structures*, 2012; **53** (4): 285-295.

[14] Van Mele T. and Block P. Algebraic graph statics. *Computer-Aided Design*, 2014; **53**: 104-116.